Reading Material

- Chapter 10, Section 6
- Chapter 11, Sections 4, 5
Weighted Graphs

- In many real-world applications, graphs have weights associated with their nodes, edges, or both.
- In this set of slides we will focus on weighted graphs where the weights are associated with edges.
Why Weighted Graphs

- Internet graph: Edge weights could correspond to the amount of data flowing along them or their bandwidth.
- Roadmap graph: Edge weights could correspond to lengths, numbers of gas stations, ...
- Food web graph: Edge weights could correspond to the total energy flow between prey and predator.
- Social network graph: Edge weights could correspond to the frequency of contact.
A weighted graph is a 3-tuple $g=(V,E,w)$, where $V$ is the set of nodes, $E$ is the set of edges, and $w:E \rightarrow R$ ($R$ is the set of reals) is a function that assigns a weight to each edge.

The definition applies to both directed and undirected graphs.
Representing Weighted Graphs

Weighted graph

0 ——— 5 ——— 1
|        |        |
|        |        |
| 2 ——— 12 ——— 3 |
Representing Weighted Graphs

Weighted graph

![Weighted graph diagram]

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<thead>
<tr>
<th>Adjacency matrix</th>
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Representing Weighted Graphs

Weighted graph

Adjacency matrix

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Adjacency list

- 0 → (1,5) (3,12)
- 1 → (0,5) (2,2)
- 2 → (1,2)
- 3 → (0,12)
A tree is a connected, acyclic graph.

A spanning tree of a graph \( g=(V,E) \) is a connected, acyclic subgraph of \( g \) that contains all the nodes in \( V \).

The weight of a spanning tree of a weighted graph \( g=(V,E,w) \) is the sum of the weights of the edges in the tree.

A minimum spanning tree (MST) of a weighted graph \( g=(V,E,w) \) is a spanning tree whose weight is minimum among all possible spanning trees of \( g \).
Paths and Distances

- Paths and connectivity are defined as in the case of unweighted graphs.
- The weight of a path is the sum of the weights of the edges on the path.
- If the weight of an edge corresponds to length, then
  - the length of a path is the sum of the lengths of the edges on the path
  - the distance between two nodes is the length of a shortest path between them
Two Central Problems on Weighted Graphs

- The **minimum-spanning tree** problem:
  - **Input:** A weighted graph $g=(V, E, w)$.
  - **Output:** The set of edges $E_T$ of an MST of $g$.

- The **single source shortest-paths** problem:
  - **Input:** A weighted graph $g=(V, E, w)$ and node $i \in V$.
  - **Output:** The distance $d_j$ from node $i$ for every node $j \in V$, and a shortest path from $i$ to $j$. 
Two Central Problems on Weighted Graphs

- It is easy to come up with brute-force algorithms for these two problems.
- However, these algorithms would be grossly inefficient.
- Efficient algorithms for both problems exist and they use an algorithmic strategy called greedy algorithms.
A greedy algorithm constructs a solution to an optimization problem through a sequence of steps, each expanding a partially constructed solution obtained so far, until a complete solution to the problem is reached.

In each step, the choice must be:

- **feasible**: it has to satisfy the problem’s constraints
- **locally optimal**: it has to be the best local choice for adding the next piece to the solution
- **irrevocable**: once made, it is not changed in subsequent steps of the algorithm.
The Minimum Spanning Tree Problem

- **Input:** A weighted graph $g=(V,E,w)$.
- **Output:** The set of edges $E_T$ of an MST of $g$. 
The Minimum Spanning Tree Problem

- Think greedy:
  - feasibility: at every point, the sub-solution must be...
  - local optimality: the algorithm chooses the next edge that ...
  - irrevocable: once an edge has been added to $E_T$, then ...
The Minimum Spanning Tree Problem: Prim’s Algorithm

**Algorithm 2: Prim.**

**Input:** Undirected, weighted graph \( g = (V, E, w) \).

**Output:** \( E_T \), the edges of an MST of \( g \).

\[ V_T \leftarrow \{x\}; \]  
// Initialize the nodes of the MST to a randomly chosen nodes \( x \)
\[ E_T \leftarrow \emptyset; \]

for \( i \leftarrow 1 \) to \( |V| - 1 \) do

- Let \( e = \{u, v\} \) be a minimum-weight edge among all edges with one endpoint in \( V_T \) and the other in \( V \setminus V_T \);
- \( V_T \leftarrow V_T \cup \{u, v\} \);
- \( E_T \leftarrow E_T \cup \{\{u, v\}\} \);

return \( E_T \);
The Minimum Spanning Tree Problem: Prim’s Algorithm

Algorithm 2: Prim.

**Input:** Undirected, weighted graph \( g = (V, E, w) \).

**Output:** \( E_T \), the edges of an MST of \( g \).

\[
V_T \leftarrow \{x\}; \quad \text{// Initialize the nodes of the MST to a randomly chosen node } x \\
E_T \leftarrow \emptyset;
\]

for \( i \leftarrow 1 \text{ to } |V| - 1 \) do

\[
\begin{align*}
&\text{Let } e = \{u, v\} \text{ be a minimum-weight edge among all edges with one endpoint in } V_T \text{ and the other in } V \setminus V_T; \\
&V_T \leftarrow V_T \cup \{u, v\}; \\
&E_T \leftarrow E_T \cup \{u, v\};
\end{align*}
\]

return \( E_T \);

What is the running time of Prim’s algorithm?
The Heap Data Structure
A heap is a binary tree (every node has 0, 1, or 2 children) with keys assigned to its nodes (one key per node) provided the following two conditions hold:

- The **tree shape requirement**: the binary tree is complete, that is, all its levels are full except possibly for the last level, where only some rightmost leaves may be missing.

- The **parental dominance requirement**: The key at each node is greater than or equal to the keys of its children.

A min-heap is one where the parental dominance requirement means the key of a node is smaller than the keys of its children.
- The root of a heap always contains its largest element.
- In a heap, a node along with all its descendants is also a heap.
- A heap can be implemented as an array by recording its elements in the top-down, left-to-right fashion. It is convenient to store the heap’s elements in positions 1 through \( n \), leaving \( H[0] \) either unused or putting there a sentinel whose value is greater than every key in the heap.
In the array representation of a heap,

- the parental node keys will be in the first \([n/2]\) positions of the array, while the leaf keys will occupy the last \([n/2]\) positions.

- The children of a key in position \(i\) will be in positions \(2i\) and \(2i+1\)
Bottom-up Heap Construction

- To construct a heap from a given list of keys, we first initialize the complete binary tree with n nodes by placing keys in the order given, and then “heapify” the tree.

- **Heapification:** Starting with the last parental node, the “heapify” algorithm checks whether the parental dominance holds for the key at this node. If it does not, the algorithm exchanges the node’s key $K$ with the larger key of its children and checks whether the parental dominance holds for $K$ in its new positions. This process continues until the parental dominance requirement for $K$ is satisfied. After completing the heapification of the subtree rooted at the current parental node, the algorithm proceeds to do the same for the node’s immediate predecessor. The algorithm stops after this is done for the tree’s root.
Bottom-up Heap Construction
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Bottom-up Heap Construction
Bottom-up Heap Construction
Bottom-up Heap Construction: The Pseudo-code

HeapBottomUp

Input: An array H[1..n] of orderable items
Modifies: Array H[1..n] is arranged in-place to reflect the heap properties
Output: None

For \( i \leftarrow \lfloor \frac{n}{2} \rfloor \) to 1

\( k \leftarrow i; \ v \leftarrow H[k]; \ heap \leftarrow false; \)

While ( (not heap) and \( 2*k \leq n \) )

\( j \leftarrow 2*k; \)

If \( j < n \)

\( \text{If } H[j] < H[j+1] \)

\( j \leftarrow j+1; \)

\( \text{If } v \geq H[j] \)

\( \text{heap} \leftarrow true; \)

Else

\( \text{H}[k] \leftarrow H[j]; \ k \leftarrow j; \)

\( \text{H}[k] \leftarrow v \)
Bottom-up Heap Construction: Computational Complexity

- Assume, for simplicity, that \( n = 2^k - 1 \) (for some natural number \( k \)).

- Let \( h \) be the height of the tree; then, \( h = \lceil \log_2 n \rceil \).

- In the worst case, each key on level \( i \) of the tree will travel to the leaf level \( h \).

- Since moving to the next level down requires two comparisons, the total number of key comparisons involving a key on level \( i \) will be \( 2(h-i) \).

- Therefore, the total number of key comparisons in the worst case will be

\[
C_{\text{worst}}(n) = \sum_{i=0}^{h-1} \sum_{\text{level } i \text{ keys}} 2(h-i) = \sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n + 1))
\]
Inserting a Key Into a Heap

- First, attach a new node with key $K$ in it after the last leaf of the existing heap.

- Then, sift $K$ up to its appropriate place in the new heap as follows.

  - Compare $K$ with its parent’s key: if the parent’s key is greater than or equal to $K$, stop; otherwise, swap these two keys and compare $K$ with its new parent.

  - This swapping continues until $K$ is not greater than its last parent or it reaches the root.
Inserting a Key Into a Heap

- First, attach a new node with key $K$ in it after the last leaf of the existing heap.

- Then, sift $K$ up to its appropriate place in the new heap as follows.

- Compare $K$ with its parent’s key: if the parent’s key is greater than or equal to $K$, stop; otherwise, swap these two keys and compare $K$ with its new parent.

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Inserting a Key Into a Heap

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❖ This swapping continues until $K$ is not greater than its last parent or it reaches the root.

![Diagram of inserting a key into a heap]

- Initial heap: 6, 8, 2, 5, 7
- Attach $K = 10$ after the last leaf (7)
- Sift $K$ up: $K$ is compared with its parent (8), then with the root (9)
- Resulting heap: 10, 9, 6, 5, 7, 2
Inserting a Key Into a Heap

- What is the running time?
Deleting the Maximum Element From a Heap

- Exchange the root’s key with the last key $K$ of the heap.
- Decrease the heap’s size by 1.
- “Heapify” the smaller tree by sifting $K$ down the tree exactly in the same way we did it in the bottom-up heap construction algorithm. That is, verify the parental dominance for $K$: if it holds, we are done; if not, swap $K$ with the larger of its children and repeat this operation until the parental dominance condition holds for $K$ in its new position.
Deleting the Maximum Element From a Heap
Deleting the Maximum Element From a Heap
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Deleting the Maximum Element From a Heap

- What is the running time?
Back to Weighted Graph Problems...
Algorithm 2: Prim.

**Input:** Undirected, weighted graph $g = (V, E, w)$.

**Output:** $E_T$, the edges of an MST of $g$.

$V_T \leftarrow \{x\}$; // Initialize the nodes of the MST to a randomly chosen node $x$

$E_T \leftarrow \emptyset$;

for $i \leftarrow 1$ to $|V| - 1$ do

  Let $e = \{u, v\}$ be a minimum-weight edge among all edges with one endpoint in $V_T$ and the other in $V \setminus V_T$;

  $V_T \leftarrow V_T \cup \{u, v\}$;

  $E_T \leftarrow E_T \cup \{\{u, v\}\}$;

return $E_T$;
The Minimum Spanning Tree Problem: Prim’s Algorithm

Algorithm 2: Prim.

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  \( V_T \leftarrow V_T \cup \{u, v\}; \)
  \( E_T \leftarrow E_T \cup \{\{u, v\}\}; \)
return \( E_T \);

What is the running time of Prim’s algorithm?
Algorithm 3: Kruskal.

**Input:** Undirected, weighted graph \( g = (V, E, w) \).

**Output:** \( E_T \), the edges of an MST of \( g \).

Sort \( E \) in non-decreasing order of the weights \( w(e_{i_1}) \leq w(e_{i_2}) \leq \cdots \leq w(e_{|E|}) \);

\[
E_T \leftarrow \emptyset;
\]

\[
\text{numedges} \leftarrow 0; \quad \text{// The number of edges in the growing MST}
\]

\[
k \leftarrow 0;
\]

while \( \text{numedges} < |V| - 1 \) do

\[
k \leftarrow k + 1;
\]

if \( E_T \cup \{e_{i_k}\} \) is acyclic then

\[
E_T \leftarrow E_T \cup \{e_{i_k}\};
\]

\[
\text{numedges} \leftarrow \text{numedges} + 1;
\]

return \( E_T \);
The Minimum Spanning Tree Problem: Kruskal’s Algorithm

Algorithm 3: Kruskal.

**Input:** Undirected, weighted graph $g = (V, E, w)$.

**Output:** $E_T$, the edges of an MST of $g$.

Sort $E$ in non-decreasing order of the weights $w(e_{i_1}) \leq w(e_{i_2}) \leq \cdots \leq w(e_{i_{|E|}})$;

$E_T \leftarrow \emptyset$;

$numedges \leftarrow 0$;

$k \leftarrow 0$;

while $numedges < |V| - 1$ do

$k \leftarrow k + 1$;

if $E_T \cup \{e_{i_k}\}$ is acyclic then

$E_T \leftarrow E_T \cup \{e_{i_k}\}$;

$numedges \leftarrow numedges + 1$;

return $E_T$;

What is the running time of Kruskal’s algorithm?
The Single-Source Shortest Paths Problem

❖ **Input:** A weighted graph $g=(V,E,w)$ and node $i \in V$.

❖ **Output:** The distance $d_j$ from node $i$ for every node $j \in V$, and a shortest path from $i$ to $j$. 
The Single-Source Shortest Paths Problem

- Think greedy:
  - feasibility: at every point, the sub-solution must be...
  - local optimality: the algorithm chooses the next node that ...
  - irrevocable: once the distance and shortest path to a node have been set, then ...
Algorithm 1: Dijkstra.

**Input:** Undirected, weighted graph $g = (V, E, w)$; source node $i$.

**Output:** Shortest paths, as well as their lengths, from $i$ to every other node in $g$.

$X \leftarrow \emptyset$;

**foreach** $j \in V$ **do**

- $d_j \leftarrow \infty$;
- $p_j \leftarrow \text{null}$;
- $X \leftarrow X \cup \{j\}$;

$d_i \leftarrow 0$;

**while** $X \neq \emptyset$ **do**

- Let $k$ be a node with the minimum value of $d_k$ in the set $X$;
- **if** $d_k = \infty$ **then**
  - break;
- $X \leftarrow X \setminus \{k\}$;

**foreach** neighbor $h$ of $k$ in the set $X$ **do**

- **if** $d_k + w((k, h)) < d_h$ **then**
  - $d_h \leftarrow d_k + w((k, h))$;
  - $p_h \leftarrow k$;

**return** $p, d;
The Single-Source Shortest Paths Problem: Dijkstra’s Algorithm

Algorithm 1: Dijkstra.

**Input**: Undirected, weighted graph $g = (V, E, w)$; source node $i$.
**Output**: Shortest paths, as well as their lengths, from $i$ to every other node in $g$.

1. $X \leftarrow \emptyset$;
2. **foreach** $j \in V$ do
   3. $d_j \leftarrow \infty$;  
      // $d_j$ is the geodesic distance between $i$ and $j$
   4. $p_j \leftarrow \text{null}$;  
      // $p_j$ carries the node label of $j$’s parent
   5. $X \leftarrow X \cup \{j\}$;
   6. $d_i \leftarrow 0$;
3. **while** $X \neq \emptyset$ do
   4. Let $k$ be a node with the minimum value of $d_k$ in the set $X$;
   5. if $d_k = \infty$ then
      6. break;
   7. $X \leftarrow X \setminus \{k\}$;
   8. **foreach** neighbor $h$ of $k$ in the set $X$ do
      9. if $d_k + w((k, h)) < d_h$ then
         10. $d_h \leftarrow d_k + w((k, h))$;
         11. $p_h \leftarrow k$;
   12. return $p, d$;

What is the running time of Dijkstra’s algorithm?
Questions?