Problems:

1. For each of the following languages, state whether each language is (I) recursive, (II) recursively enumerable but not recursive, or (III) not recursively enumerable. Prove your answer.

- $L_1 = \{\langle M \rangle | M \text{ is a TM and there exists an input on which } M \text{ halts in less than } |\langle M \rangle| \text{ steps} \}.$
- $L_2 = \{\langle M \rangle | M \text{ is a TM and } |L(M)| \leq 3 \}.$
- $L_3 = \{\langle M \rangle | M \text{ is a TM and } |L(M)| \geq 3 \}.$
- $L_4 = \{\langle M \rangle | M \text{ is a TM that accepts all even numbers} \}.$
- $L_5 = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is finite} \}.$
- $L_6 = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is infinite} \}.$
- $L_7 = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is countable} \}.$
- $L_8 = \{\langle M \rangle | M \text{ is a TM and } L(M) \text{ is uncountable} \}.$
- $L_9 = \{\langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \cup L(M_2) \}.$
- $L_{10} = \{\langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \cap L(M_2) \}.$
- $L_{11} = \{\langle M_1, M_2 \rangle | M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \setminus L(M_2) \}.$
- $L_{12} = \{\langle M \rangle | M \text{ is a TM, } M_0 \text{ is a TM that halts on all inputs, and } M_0 \in L(M) \}.$
- $L_{13} = \{\langle M \rangle | M \text{ is a TM, } M_0 \text{ is a TM that halts on all inputs, and } M \in L(M_0) \}.$
- $L_{14} = \{\langle M, x \rangle | M \text{ is a TM, } x \text{ is a string, and there exists a TM, } M', \text{ such that } x \notin L(M) \cap L(M') \}.$
- $L_{15} = \{\langle M \rangle | M \text{ is a TM, and there exists an input on which } M \text{ halts within 1000 steps} \}.$
- $L_{16} = \{\langle M \rangle | M \text{ is a TM, and there exists an input whose length is less than 100, on which } M \text{ halts} \}.$
- $L_{17} = \{\langle M \rangle | M \text{ is a TM, and } M \text{ is the only TM that accepts } L(M) \}.$
- $L_{18} = \{\langle k, x, M_1, M_2, \ldots, M_k \rangle | k \text{ is a natural number, } x \text{ is a string, } M_i \text{ is a TM for all } 1 \leq i \leq k, \text{ and at least } k/2 \text{ TMs of } M_1, \ldots, M_k \text{ halt on } x \}.$
- $L_{19} = \{\langle M \rangle | M \text{ is a TM, and } |M| < 1000 \}.$
- $L_{20} = \{\langle M \rangle | \exists x, |x| \equiv_5 1, \text{ and } x \in L(M) \}.$
- $L_{21} = \{\langle M \rangle | M \text{ is a TM, and } M \text{ halts on all palindromes} \}.$
- $L_{22} = \{\langle M \rangle | M \text{ is a TM, and } L(M) \cap \{a^n | n \geq 0 \} \text{ is empty} \}.$
- $L_{23} = \{\langle M, k \rangle | M \text{ is a TM, and } |\{w \in L(M) : w \in a^*b^*\}| \geq k \}.$
- $L_{24} = \{\langle M \rangle | M \text{ is a TM that halts on all inputs and } L(M) = L' \text{ for some undecidable language } L' \}.$
- $L_{25} = \{\langle M \rangle | M \text{ is a TM, and } M \text{ accepts (at least) two strings of different lengths} \}.$
- $L_{26} = \{\langle M \rangle | M \text{ is a TM such that both } L(M) \text{ and } L(M) \text{ are infinite} \}.$
- $L_{27} = \{\langle M, x, k \rangle | M \text{ is a TM, and } M \text{ does not halt on } x \text{ within } k \text{ steps} \}.$
- $L_{28} = \{\langle M \rangle | M \text{ is a TM, and } |L(M)| \text{ is prime} \}.$
- $L_{29} = \{\langle M \rangle | \text{ there exists } x \in \Sigma^* \text{ such that for every } y \in L(M), xy \notin L(M) \}.$
- $L_{30} = \{\langle M \rangle | \text{ there exist } x, y \in \Sigma^* \text{ such that either } x \in L(M) \text{ or } y \notin L(M) \}.$
- $L_{31} = \{\langle M \rangle | \text{ there exists a TM } M' \text{ such that } \langle M \rangle \neq \langle M' \rangle \text{ and } L(M) = L(M') \}.$
- $L_{32} = \{\langle M_1, M_2 \rangle | L(M_1) \preceq_m L(M_2) \}.$
- $L_{33} = \{ \langle M \rangle | M \text{ does not accept any string } w \text{ such that } 001 \text{ is a prefix of } w \}$.
- $L_{34} = \{ \langle M, x \rangle | M \text{ does not accept any string } w \text{ such that } x \text{ is a prefix of } w \}$.
- $L_{35} = \{ \langle M, x \rangle | x \text{ is prefix of } M \}$.
- $L_{36} = \{ \langle M_1, M_2, M_3 \rangle | L(M_1) = L(M_2) \cup L(M_3) \}$.
- $L_{37} = \{ \langle M_1, M_2, M_3 \rangle | L(M_1) \subseteq L(M_2) \cup L(M_3) \}$.
- $L_{38} = \{ \langle M \rangle | \text{there exist two TMs } M_2 \text{ and } M_3 \text{ such that } L(M_1) \subseteq L(M_2) \cup L(M_3) \}$.
- $L_{39} = \{ \langle M, w \rangle | M \text{ is a TM that accepts } w \text{ using at most } 2|w| \text{ squares of its tape} \}$.

2. If $A \leq_m B$ and $B$ is a regular language, does that imply that $A$ is a regular language?

3. Recall the language $A_{TM} = \{ \langle M, w \rangle | M \text{ is a TM, and } M \text{ accepts } w \}$. Consider the language

$$J = \{ w | w = 0x \text{ for some } x \in A_{TM} \text{ or } w = 1y \text{ for some } y \in \overline{A_{TM}} \}.$$  

   (a) Show that $J$ is not in RE.
   (b) Show that $\overline{J}$ is not in RE.
   (c) Show that $J \leq_m \overline{J}$.

4. Show that if a language $A$ is in RE and $A \leq_m \overline{A}$, then $A$ is recursive.

5. A language $L$ is **RE-Complete** if:

   - $L \in RE$, and
   - $L' \leq_m L$ for all $L' \in RE$.

Recall the following languages:

$$L_{\Sigma^*} = \{ \langle M \rangle | L(M) = \Sigma^* \}$$

$$HP = \{ \langle M, w \rangle | M \text{ halts on } w \}$$

   (a) Is $L_{\Sigma^*}$ RE-Complete or not? Prove your answer.
   (b) Is $HP$ RE-Complete or not? Prove your answer.

6. Let $L_1$, $L_2$ be two decidable languages, and let $L$ be a language such that $L_1 \subseteq L \subseteq L_2$. Is $L$ decidable or not? Prove your answer.

7. Let $L$ be a language RE. Show that $L' = \{ x | \exists y : (x, y) \in L \}$ is also RE.

8. Prove or disprove: there exists an undecidable unary language (a unary language is a subset of $1^*$).

9. **Problem Formulation**.

   (a) Consider the problem of testing whether a TM $M$ on an input $w$ ever attempts to move its head left when its head is on the leftmost tape cell. Formulate this problem as a language and show that it is undecidable.

   (b) Consider the problem of testing whether a TM $M$ on an input $w$ ever attempts to move its head left at any point during its computation on $w$. Formulate this problem as a language and show that it is decidable.

10. Let $A$ and $B$ be two disjoint languages. We say that language $C$ **separates** $A$ and $B$ if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-RE languages are separable by some decidable language.

11. Suppose there are four languages $A$, $B$, $C$, and $D$. Each of the languages may or may not be recursively enumerable. However, we know the following about them:
There is a reduction from $A$ to $B$.
There is a reduction from $B$ to $C$.
There is a reduction from $D$ to $C$.

Below are four statements. Indicate whether each one is
(a) CERTAIN to be true, regardless of what problems $A$ through $D$ are.
(b) MAYBE true, depending on what $A$ through $D$ are.
(c) NEVER true, regardless of what $A$ through $D$ are.

Please, justify your answer!

(a) $A$ is recursively enumerable but not recursive, and $C$ is recursive.

(b) $A$ is not recursive, and $D$ is not recursively enumerable.

(c) If $C$ is recursive, then the complement of $D$ is recursive.

(d) If $C$ is recursively enumerable, then $B \setminus D$ is recursively enumerable.

12. Recall the following definition: A grammar $G$ computes a function $f$ iff for all $u, v \in \Sigma^*$,

$$SuS \Rightarrow_G^* v \text{ iff } f(u) = v.$$ 

For each of the following functions, show a grammar that computes it. In the functions $f_1, \ldots, f_4$, both $n$ and $f(n)$ are unary representations of natural numbers. For functions $f_5, \ldots, f_8$, the input/output alphabet is specified.

- $f_1(n) = 3n + 5.$
- $f_2(n) = \begin{cases} 
1 & \text{if } n \equiv 0 \pmod{3} \\
11 & \text{if } n \equiv 1 \pmod{3} \\
111 & \text{if } n \equiv 2 \pmod{3} 
\end{cases}$
- $f_3(n) = n - 1.$
- $f_4(n) = n/2.$ Assume $n$ is even.
- $f_5(w) = ww$, where $w \in \{a, b\}^*$.
- $f_6 = w'$, where $w \in \{a, b\}^*$, and $w'$ is obtained from $w$ by replacing the $a$'s by $b$'s and $b$'s by $a$'s. For example, $f_6(aaba) = bbab$.
- $f_7(a_1 a_2 \ldots a_k) = a_1 a_1 a_2 a_2 \ldots a_k a_k$, where each $a_i$ is in the alphabet $\{a, b\}$. For example, $f_7(aaba) = aaaaabaa$.
- $f_8(w) = \begin{cases} 
f_5(w) & \text{if the rightmost symbol of } w \text{ is } a \\
f_7(w) & \text{if the rightmost symbol of } w \text{ is } b 
\end{cases}$ ($\Sigma = \{a, b\}$).

13. Show that the following languages are recursive.

- $L_{40} = \{\langle M \rangle | M \text{ is a DFA and } L(M) \text{ is finite} \}$.
- $L_{41} = \{\langle M \rangle | M \text{ is a DFA and } L(M) = \Sigma^* \}$.
- $L_{42} = \{\langle M, x \rangle | M \text{ is a DFA and } M \text{ accepts } x \}$.
- $L_{43} = \{\langle M, x \rangle | M \text{ is a DFA and } M \text{ halts on } x \}$.
- $L_{44} = \{\langle G \rangle | G \text{ is a CFG and } L(G) = \emptyset \}$.
- $L_{45} = \{\langle M \rangle | M \text{ is a DFA and } M \text{ accepts some string of the form } ww^R \text{ for some } w \in \{a, b\}^* \}$.

14. Prove that each of the following languages are not context-free, and write unrestricted grammars that generate them.
\[ L_{46} = \{ x \# w \mid x, w \in \{a, b\}^* \text{ and } x \text{ is a substring of } w \}. \]
\[ L_{47} = \{ w \in \{a, b, c\}^* \mid \#_a(w) \geq \#_b(w) \geq \#_c(w) \}. \]
\[ L_{48} = \{ a^n b^n c^n a^n b^n \mid n > 0 \}. \]
\[ L_{49} = \{ a^n b^n c^n a^n b^n c^n \mid n \geq 0 \}. \]
\[ L_{50} = \{ a^n b^n c^n d^n m \mid m, n \geq 0 \}. \]
\[ L_{51} = \{ w \in \{1\}^* \mid w \text{ is the unary encoding of } 2^k \text{ for some } k > 0 \}. \]

15. Let \( L_{52} \) be the language containing only the single string \( s \), where
\[
    s = \begin{cases} 
    0 & \text{if God does not exist} \\
    1 & \text{if God exists} 
    \end{cases}
\]

Is \( L_{52} \) decidable? Why or why not? (Note that the answer does not depend on your religious convictions.)