Homework 1 – Due Thursday, September 10, 2015

Please refer to the course website for the full homework policy and options.

Reminders

• Your solutions are due in class. Please also submit a soft copy on the dropbox on Owlspace. Late homework will not be accepted.

• Collaboration is permitted, but see the collaboration policies on the course website. *Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.*

• To facilitate grading, please write down your solution to each problem on a separate sheet of paper. Make sure to include all identifying information and your collaborators on each sheet.

• Some of the questions below ask you to write functions and/or data definitions. It is acceptable to write these functions in mathematically rigorous English, or as pseudocode, or in Dafny syntax. You may find the last option the most useful, as you can test the correctness of your functions.

• When in doubt, email the TAs or the instructor.

Practice problems

These problems are provided just for practice, and should not be handed in.

1. Functional programming:

   (a) Write a function that returns the number of occurrences of a given number in an (inductively defined) list of numbers.

   (b) Write a function `merge` that, given two sorted lists \( L_1 \) and \( L_2 \) of numbers, returns a sorted list containing all the elements of \( L_1 \) and \( L_2 \), and nothing else.

   (c) Write a function `swapper` that takes three arguments — integers \( p \) and \( q \), and a list of integers \( L \) — and returns a list that is the same as \( L \), but has all occurrences of \( p \) replaced by \( q \).

   (d) Write a function to count the number of leaves in a binary tree.

2. Inductive proofs:

   (a) Prove that \( n^2 - 1 \) is divisible by 8 for all odd positive integers \( n \).

   (b) Prove that \( 1 + nh \leq (1 + h)^n \) for \( n \geq 0 \), where \( h > -1 \).

   (c) Show that every positive integer \( n \) can be written as a sum of distinct powers of two, that is, as a sum of the integers \( 2^0 = 1, 2^1 = 2, 2^2 = 4 \), and so on.

   (d) Show that the sum of the interior angles of a convex \( n \)-gon is \( 180(n - 2) \) degrees.
(e) (Towers of Hanoi) Suppose you have three posts and a stack of \( n \) disks, initially placed on one post, with the largest disk on the bottom and each disk above it being smaller than the disk immediately below. A legal move involves taking the top disk from one post and moving it so that it becomes the top disk on another post; however, every move must place a disk either on an empty post, or on top of a disk larger than it. Show that for every \( n \) there is a sequence of moves that will terminate with all the disks on a post different from the original one. How many moves are needed for an initial stack of \( n \) disks?

Problems to be handed in

1. Assume that a chocolate bar consists of \( n \) squares arranged in a rectangular pattern. The entire bar, or a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Determine the minimum number \( b(n) \) (naturally, this number depends on \( n \)) of breaks you must successively make to break the bar into \( n \) separate squares. Use induction to show that the bar can in fact be broken down into squares with \( b(n) \) breaks.

2. What is wrong with the following proof that all cows have the same color?

   Let \( P(n) \) be the property that all cows in a set of \( n \geq 1 \) cows have the same color.
   
   **Base case:** Clearly, \( P(1) \) is true.
   
   **Inductive case:** Now assume that \( P(n) \) is true for an arbitrary \( n \). That is, assume that all cows in any set of \( n \) cows are the same color. Consider any \( n + 1 \) cows; number them 1, 2, \ldots, \( n + 1 \). Now, by the inductive hypothesis, the first \( n \) of these cows all must have the same color, and the last \( n \) of them must also have the same color. Since the set of the first \( n \) cows overlaps with the set of the last \( n \) cows, all \( (n + 1) \) cows must be of the same color. This shows that \( P(n + 1) \) is true and finishes the proof by induction.

3. Recall that the \( n \)-th Fibonacci number is defined by the following equations:

   \[
   \begin{align*}
   Fib(0) &= 0 \\
   Fib(1) &= 1 \\
   Fib(n) &= Fib(n - 1) + Fib(n - 2) & \text{for } n \geq 2.
   \end{align*}
   \]

   Show that \( Fib(n) < 2^n \) for all \( n \).

4. This problem asks you to write functions over the inductively defined lists and trees. Each function should be accompanied by a proof of correctness.

   (a) Write a function to pack consecutive duplicates of list elements into separate sublists. In other words, on input \([1; 1; 1; 2; 3; 1; 1]\), your function should output the following list of lists: \([[[1; 1]; [2]; [3]; [1; 1]]]\).

   (b) Write a function to *insert* an element \( x \) into a sorted list, so the resulting list is also sorted.

   (c) Write a function that takes a tree \( T \) (whose nodes contain integer values) and an integer \( n \), and returns the tree that is obtained by deleting all subtrees rooted at nodes \( u \) in \( T \) containing the value \( n \).

5. The set of well-formed boolean expressions (wfbe) over a class of boolean variables is defined as follows:

   - *True* is a wfbe; *False* is a wfbe
• A boolean variable is a wfbe
• If $F$ is a wfbe, then ($\neg F$) is also a wfbe
• If $F$ and $G$ are wfbes, then ($F \land G$) and ($F \lor G$) are wfbes

A wfbe is in \textit{negation normal form} if:

• Every negation in the wfbe is applied to a variable, not to a more complicated subformula
• Either the entire expression is \textit{True}, or the expression does not contain \textit{True}
• Either the entire expression is \textit{False}, or the expression does not contain \textit{False}.

Prove that for every wfbe, there is a logically equivalent wfbe in negation normal form. For example, the wfbe
\[
(\neg((p \land q) \lor \neg r)) \land (\neg(p \lor \neg r) \land q)
\]
where $p$, $q$, and $r$ are boolean variables, is logically equivalent to the following wfbe in negation normal form:
\[
(((\neg p \lor \neg q) \land r) \land ((\neg p \land r) \land q).
\]

6. Suppose you have a bag of red, yellow, and blue chips; the number of chips of each kind in the bag is unknown. Consider a process where you repeatedly perform the following steps.

If only one chip remains in the bag, you take it out. Otherwise, you remove two chips at random.

• If one of the two removed chips is red, you do not put any chips in the bag.
• If both removed chips are yellow, you put one yellow chip and five blue chips in the bag.
• If one of the chips is blue and the other is not red, you put 10 red chips in the bag.

Prove using induction that this process always halts.