Practice problem set on NP-completeness

This problem set is for practice, and will not be graded. A subset of these problems will be discussed in class on Thursday, December 4. However, we encourage you to try solving as many of these problems yourself as possible.

1. (True or false) For each of these questions below, decide whether the answer is “Yes”, “No”, or “Unknown, as it would resolve the question $P = NP$.

   (a) Let’s define the decision version of the Interval Scheduling Problem (studied in the context of greedy algorithms): Given a collection of intervals on a time-line, and a bound $k$, does the collection contain a subset of nonoverlapping intervals of size at least $k$?
   Question: Is it the case that Interval Scheduling $\leq_P$ Vertex Cover?

   (b) Question: Is it the case that Independent Set $\leq_P$ Interval Scheduling?

2. (Efficient recruiting) Suppose you are helping organize a summer camp. The camp is supposed to have at least one counselor who is skilled at each of the $n$ sports covered by the camp (baseball, volleyball, and so on). They have received job applicants from $m$ potential counselors. For each of the $n$ sports, there is a subset of the $m$ applicants qualified in that sport. The question is: For a given number $k < m$, is it possible to hire at most $k$ of the counselors and have at least one counselor qualified in each of the $k$ sports? We’ll call this the Efficient Recruiting Problem.

   Show that Efficient Recruiting is NP-complete.

3. (Zero-weight-cycle) You are given a directed graph $G = (V, E)$ with weights $w_e$ on its edges $e \in E$. The weights can be negative or positive. The Zero-Weight-Cycle Problem is to decide if there is a simple cycle in $G$ so that the sum of the edge weights in this cycle is exactly 0. Prove that this problem is NP-complete.

4. (Plot fulfilment) Hypertext fiction is a type of fiction consisting of a set of pages, each containing some text, along with links between them. The reader of such a hypertext story follows a trail — a sequence of pages that begins with a start page $s$ and finishes at an end page $t$ — through the story.

   Suppose you have created a piece of hypertext fiction, and you will like to know if there is a trail through your story that contains each of a set of thematic elements. You have $n$ thematic elements, the $i$-th of which can be identified with a set $T_i$ of pages. You formalize this as the Plot Fulfilment problem, which asks: is there a trail from $s$ to $t$ that contains at least one page from each of the sets $T_i$?

   Show that this problem is NP-complete.

5. (Feedback set) Given an undirected graph $G = (V, E)$, a feedback set is a set $X \subseteq V$ with the property that $G - X$ has no cycles. The Undirected Feedback Set Problem asks: Given $G$ and $k$, does $G$ contain a feedback set of size at most $k$? Prove that Undirected Feedback Set is NP-complete.
6. (Monotone Satisfiability) Consider the Monotone Satisfiability Problem, which is exactly like the CNF satisfiability problem, except all variables appearing in a clause have to be non-negated (i.e., clauses cannot have occurrences of literals \( \lnot \)). Any instance of this problem is trivially satisfiable; you could just set each variable to true.

However, consider the variant of this problem, called Monotone Satisfiability with Few True Variables, which asks: Is there a satisfying assignment for the instance in which at most \( k \) variables are set to true. Show that this problem is NP-complete.

7. (Number partitioning) You are given a set of positive integers \( \{x_1, \ldots, x_n\} \); you want to decide whether the numbers can be divided into two sets \( S_1 \) and \( S_2 \) with the same sum:

\[
\sum_{x \in S_1} x = \sum_{x \in S_2} x
\]

Show that this problem is NP-complete.