A connection between lambda calculus and maps

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1. Dreaming of purely functional sheep

One night you have trouble falling asleep (after watching a disturbing documentary about mutable states), and so you decide to start counting terms of pure lambda calculus. Since there are infinitely many, to not get lost you group terms by their total number of variable occurrences, and to avoid double-counting you count only the β-normal terms. Since this is still pretty hard to keep track of, you decide to limit your counting to terms which are *linear* (in the sense that every variable is used exactly once) and *planar* (in the sense that there are no free variables occurring between a variable and its lambda binder). With these constraints and your eyes closed, you count one term of size one,

$$\lambda x.x$$

two terms of size two,

$$\lambda x.x(\lambda y.y) \qquad \lambda x.\lambda y.yx$$

and nine terms of size three:

$$\lambda x.x(\lambda y.y(\lambda z.z)) \qquad \lambda x.x(\lambda y.\lambda z.zy) \qquad \lambda x.(x(\lambda y.y))(\lambda z.z) \qquad \lambda x.\lambda y.y(x(\lambda z.z))$$

$$\lambda x.\lambda y.y(\lambda z.zx) \qquad \lambda x.\lambda y.(y(\lambda z.z))x \qquad \lambda x.\lambda y.(yx)(\lambda z.z) \qquad \lambda x.\lambda y.\lambda z.z(yx) \qquad \lambda x.\lambda y.\lambda z.(zy)x$$

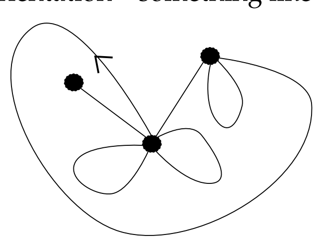
At this point you are still wide awake, and so you carry on for a while. After what seems like hours, you finally reach for the smartphone by the side of your bed (knowing that it won't help your insomnia but unable to resist) and navigate the web browser to the Online Encyclopedia of Integer Sequences [1]. Squinting into the glare of the phone while trying to type with reasonable accuracy, you enter the following list of numbers:

$$1, 2, 9, 54, 378, 2916, 24057$$

To your surprise, the Encyclopedia returns a perfect match, a certain entry named "A000168". You click on the link to that entry and start reading. Your skin begins to crawl. Suddenly you hear footsteps just outside your bedroom, and a door creaking open...

2. The correspondence

Apparently, there is a bijective correspondence between normal planar lambda terms and *rooted planar maps* [3]. A rooted planar map is essentially a connected graph embedded on the plane (or equivalently on a sphere), with one edge marked and assigned an orientation—something like so:

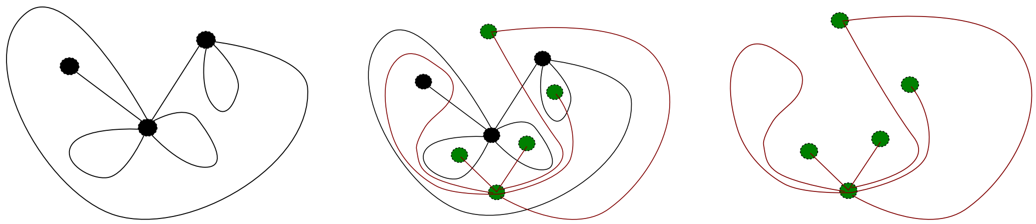


At first sight, it might not be obvious why such a thing should correspond to a lambda term, but it is possible to establish a bijection by showing how Tutte's analysis of rooted planar maps [2] (decomposing them into the vertex map, maps with an isthmic root, and maps with a non-isthmic root) may be replayed in lambda calculus. It is still not clear, though, whether this is just a big coincidence or an indication of something more meaningful.

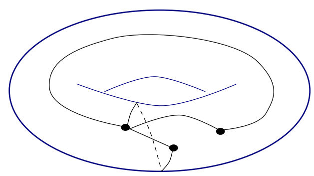
3. A few open questions

If the correspondence really means something, then it should be possible to relate the different features of lambda calculus to the theory of maps, and vice versa. Concentrating on the reverse direction, we pose a few unconventional questions about lambda calculus:

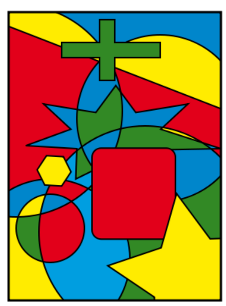
1. Originally, rooted maps were considered as a technical device to aid in counting, since unrooted maps can have non-trivial symmetries. **What (if anything) does it mean to "unroot" a lambda term?**
2. Swapping faces with vertices is a dualizing operation on maps:

   What (if anything) is the meaning of face-vertex duality in lambda calculus?
3. In general, a map need not be planar—one can consider graphs embedded on surfaces of arbitrary genus, for example on a torus:

   Is there a natural notion of genus for lambda terms?
4. (Unrooted) planar maps are the subject of the Four Color Theorem:[1]

   Does the four color theorem tell us anything about planar lambda terms?

References

[1] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A000168`.

[2] W. T. Tutte. On the enumeration of planar maps. *Bulletin of the American Mathematical Society*, 74:64–74, 1968.

[3] Noam Zeilberger and Alain Giorgetti. A correspondence between rooted planar maps and normal planar lambda terms. August 21, 2014. Under submission, draft manuscript available at `http://arxiv.org/abs/1408.5028`.

[1] Image credit: Inductiveload, via Wikimedia Commons.