Formulas: Lowercase Greek letters.

Sentences: σ, τ.

Sets of formulas: Uppercase Greek letters, plus certain italic letters which pretend to be Greek, viz., A (alpha) and T (tau).

Structures: Uppercase German letters.

EXERCISES

1. Assume that we have a language with the following parameters: $\forall$, intended to mean "for all things"; N, intended to mean "is a number"; I, intended to mean "is interesting"; $<$, intended to mean "is less than"; and $\mathbf{0}$, a constant symbol intended to denote zero. Translate into this language the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.

(a) Zero is less than any number.

(b) If any number is interesting, then zero is interesting.

(c) No number is less than zero.

(d) Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.

(e) There is no number such that all numbers are less than it.

(f) There is no number such that no number is less than it.

In 2–6, translate each English sentence into the first-order language specified. Make full use of the notational conventions and abbreviations to make the end result as readable as possible.

2. Neither a nor b is a member of every set. ($\forall$, for all sets; $\in$, is a member of; a, a; b, b.)

3. If horses are animals, then heads of horses are heads of animals. ($\forall$, for all things; E, is a horse; A, is an animal; hx, the head of x or (if x is headless) x itself.)

4. (a) You can fool some of the people all of the time. (b) You can fool all of the people some of the time. (c) You can't fool all of the people all of the time. ($\forall$, for all things; P, is a person; T, is a time; Fxy, you can fool x at y. One or more of the above may be ambiguous, in which case you will need more than one translation.)

5. (a) Adams can't do every job right. (b) Adams can't do any job right. ($\forall$, for all things; J, is a job; a, Adams; Dxy, x can do y right.)

6. Nobody likes everybody. ($\forall$, for all people; Lxy, x likes y.)

This corollary is sometimes useful in showing that a given relation is *not* definable. Consider, for example, the structure $(\mathbb{R}, <)$ consisting of the set of real numbers with its usual ordering. An automorphism of this structure is simply a function h from $\mathbb{R}$ onto $\mathbb{R}$ which is strictly increasing:

$$a < b \Leftrightarrow h(a) < h(b).$$

One such automorphism is the function h for which $h(a) = a^3$. Since this function maps points outside of $\mathbb{N}$ into $\mathbb{N}$, the set $\mathbb{N}$ is not definable in this structure.

Another example is provided by elementary algebra books, which sometimes explain that the length of a vector in the plane cannot be defined in terms of vector addition and scalar multiplication. For the map which takes a vector $\boldsymbol{x}$ into the vector $2\boldsymbol{x}$ is an automorphism of the plane with respect to vector addition and scalar multiplication, but it is not length-preserving. From our viewpoint, the structure in question,

$$(E, +, f_r)_{r \in \mathbb{R}},$$

has for its universe the plane E, has the binary function $+$ of vector addition, and has (for each r in the set $\mathbb{R}$) the unary function f_r of scalar multiplication by r. (Thus the language in question has a one-place function symbol for each real number.) The doubling map described above is an automorphism of this structure. But it does not preserve the set of unit vectors,

$$\{\boldsymbol{x} : \boldsymbol{x} \in E \text{ and } \boldsymbol{x} \text{ has length } 1\}.$$

So this set cannot be definable in the structure. (Incidentally, the homomorphisms of vector spaces are normally called *linear transformations*.)

EXERCISES

1. Show that (a) $\Gamma ; \alpha \models \varphi$ iff $\Gamma \models (\alpha \to \varphi)$; and (b) $\varphi \models\!\!\!\dashv \psi$ iff $\models (\varphi \leftrightarrow \psi)$.

2. Show that no one of the following sentences is logically implied by the other two. (This is done by giving a structure in which the sentence in question is false, while the other two are true.)

(a) $\forall x \forall y \forall z (Pxy \to Pyz \to Pxz)$. Recall that by our convention $\alpha \to \beta \to \gamma$ is $\alpha \to (\beta \to \gamma)$.
(b) $\forall x \forall y (Pxy \to Pyx \to x \approx y)$.
(c) $\forall x \exists y\, Pxy \to \exists y \forall x\, Pxy$.

3. Show that
$$\{\forall x(\alpha \to \beta), \forall x\, \alpha\} \models \forall x\, \beta.$$

4. Show that if x does not occur free in α, then $\alpha \models \forall x\, \alpha$.

5. Show that the formula $x \approx y \to Pzfx \to Pzfy$ (where f is a one-place function symbol and P is a two-place predicate symbol) is valid.

6. Show that a formula θ is valid iff $\forall x\, \theta$ is valid.

7. Restate the definition of "$\mathfrak{A}$ satisfies φ with s" in the way described on pages 82f. That is, define by recursion a function $\bar{h}$ such that $\mathfrak{A}$ satisfies φ with s iff $s \in \bar{h}(\varphi)$.

8. Assume that Σ is a set of sentences such that for any sentence τ, either $\Sigma \models \tau$ or $\Sigma \models \neg\tau$. Assume that $\mathfrak{A}$ is a model of Σ. Show that for any sentence τ, $\models_{\mathfrak{A}} \tau$ iff $\Sigma \models \tau$.

9. Assume that the language has equality and a two-place predicate symbol P. For each of the following conditions, find a sentence σ such that the structure $\mathfrak{A}$ $(= (|\mathfrak{A}|, P^{\mathfrak{A}}))$ is a model of σ iff the condition is met.

(a) $|\mathfrak{A}|$ has exactly two members.
(b) $P^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ into $|\mathfrak{A}|$.
(c) $P^{\mathfrak{A}}$ is a permutation of $|\mathfrak{A}|$; i.e., $P^{\mathfrak{A}}$ is a one-to-one function with domain and range equal to $|\mathfrak{A}|$.

10. Show that
$$\models_{\mathfrak{A}} \forall \mathbf{v}_2\, Q\mathbf{v}_1\mathbf{v}_2 [\![c^{\mathfrak{A}}]\!] \qquad \text{iff } \models_{\mathfrak{A}} \forall \mathbf{v}_2\, Qc\mathbf{v}_2.$$

Here Q is a two-place predicate symbol and c is a constant symbol.

11. For each of the following relations, give a formula which defines it in $(\mathbb{N}, +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$.)

(a) $\{0\}$.
(b) $\{1\}$.
(c) $\{\langle m, n\rangle : n$ is the successor of m in $\mathbb{N}\}$.
(d) $\{\langle m, n\rangle : m < n$ in $\mathbb{N}\}$.

12. Let $\mathfrak{R}$ be the structure $(\mathbb{R}, +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$. $\mathfrak{R}$ is the structure whose universe is the set $\mathbb{R}$ of real numbers and such that $+^{\mathfrak{R}}$ and $\cdot^{\mathfrak{R}}$ are the usual addition and multiplication operations.)

(a) Give a formula which defines in $\mathfrak{R}$ the set $[0, \infty)$.

(b) Give a formula which defines in $\mathfrak{R}$ the set $\{2\}$.

*(c) Show that any finite union of intervals, the endpoints of which are algebraic, is definable in $\mathfrak{R}$. (The converse is also true; these are the only definable sets in the structure. But we will not prove this fact.)

13. Prove part (a) of the homomorphism theorem.

14. What subsets of the real line $\mathbb{R}$ are definable in $(\mathbb{R}, <)$? What subsets of the plane $\mathbb{R} \times \mathbb{R}$ are definable in $(\mathbb{R}, <)$?

15. Show that the addition relation, $\{\langle m, n, p\rangle : p = m + n\}$, is not definable in $(\mathbb{N}, \cdot)$. *Suggestion*: Consider an automorphism of $(\mathbb{N}, \cdot)$ which switches two primes.

16. Let $\mathfrak{A}$ be a structure; let B be a set which includes $|\mathfrak{A}|$. Show that there is a structure $\mathfrak{B}$ whose universe is B and such that for any sentence σ not containing $\approx$, $\models_{\mathfrak{A}} \sigma$ iff $\models_{\mathfrak{B}} \sigma$. *Suggestion*: Choose some $a_0 \in |\mathfrak{A}|$. Let $h : B \to |\mathfrak{A}|$ map points in $|\mathfrak{A}|$ into themselves and map other points of B into a_0. Define $\mathfrak{B}$ so that h is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$.

17. (a) Consider a language with equality whose only parameter (aside from $\forall$) is a two-place predicate symbol P. Show that if $\mathfrak{A}$ is finite and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

*(b) Show that the result of part (a) holds regardless of what parameters the language contains.

18. A universal ($\forall_1$) formula is one of the form $\forall x_1 \cdots \forall x_n\, \theta$, where θ is quantifier-free. An existential ($\exists_1$) formula is of the dual form $\exists x_1 \cdots \exists x_n\, \theta$. Let $\mathfrak{A}$ be a substructure of $\mathfrak{B}$, and let $s : V \to |\mathfrak{A}|$.

(a) Show that if $\models_{\mathfrak{B}} \varphi\,[s]$ and φ is universal, then $\models_{\mathfrak{A}} \varphi\,[s]$. And if $\models_{\mathfrak{A}} \psi\,[s]$ and ψ is existential, then $\models_{\mathfrak{B}} \psi\,[s]$.

(b) Conclude that the sentence $\forall x\, Px$ is not logically equivalent to any existential sentence, nor $\exists x\, Px$ to any universal sentence.

19. An $\exists_2$ formula is one of the form $\exists x_1 \cdots \exists x_n\, \theta$, where θ is universal.

(a) Show that if an $\exists_2$ sentence not containing function symbols is true in $\mathfrak{A}$, then it is true in some finite substructure of $\mathfrak{A}$.

(b) Conclude that $\forall x\, \exists y\, Pxy$ is not logically equivalent to any $\exists_2$ sentence.

20. Assume the language has equality and a two-place predicate symbol P. Consider the two structures $(\mathbb{N}, <)$ and $(\mathbb{R}, <)$ for the language.