Propositional Semantics:

1. Implement a truth evaluator $\text{eval}(\varphi, \tau)$, which evaluates whether a formula $\varphi$ holds for a truth assignment $\tau$. Use the evaluator to answer the following question:

Let $\alpha$ be the formula:

$$(p_1 \rightarrow (p_2 \land p_3)) \land ((\neg p_1) \rightarrow (p_3 \land p_4)).$$

Let $\beta$ be the formula:

$$(p_3 \rightarrow (\neg p_6)) \land ((\neg p_3) \rightarrow (p_4 \rightarrow p_1)).$$

Let $\gamma$ be the formula:

$$(\neg (p_2 \land p_5)) \land (p_2 \rightarrow p_6))$$

Let $\delta$ be the formula:

$$(-p_3 \rightarrow p_6))$$

Evaluate the formula $\psi$

$$((\alpha \land (\beta \land \gamma)) \rightarrow \delta)$$

under the truth assignment $I_1$, where $I_1(p_1) = I_1(p_3) = I_1(p_5) = 0$ and $I_1(p_2) = I_1(p_4) = I_1(p_6) = 1$, as well as under the truth assignment $I_2$, where $I_2(p_1) = I_2(p_3) = I_2(p_5) = 1$ and $I_2(p_2) = I_2(p_4) = I_2(p_6) = 0.$
Validity:

1. Prove the validity of the following principles:
   
   (a) \(((p \rightarrow q) \rightarrow p) \rightarrow p\) - Peirce’s Law
   
   (b) \(false \rightarrow p\) - Ex Falso Quolibet (Here false is a formula that is always false.)
   
   (c) \(((\neg p) \lor p)\) : Excluded Middle (“Tertium Non Datur”)
   
   (d) \(((p \rightarrow q) \lor p)\): Weak Excluded Middle

2. Prove Disjunctive Propositional Completeness: if \(\models (\alpha \lor \beta)\) and \(AP(\alpha) \cap AP(\beta) = \emptyset\), then \(\models \alpha\) or \(\models \beta\)

Logical Implication:

1. A binary connective \(\circ\) is \textit{commutative} if \((\theta \circ \psi)\) is logically equivalent to \((\psi \circ \theta)\); it is \textit{associative} if \(((\varphi \circ \theta) \circ \psi)\) is logically equivalent to \((\varphi \circ (\theta \circ \psi))\).

   Analyze the binary connectives \(\land, \lor, \rightarrow,\) and \(\leftrightarrow\) to see whether they are commutative and associative. Prove your claims.

2. Prove de Morgan’s Laws
   
   \(\models (\neg (p \land q)) \leftrightarrow ((\neg p) \lor (\neg q)))\)
   
   \(\models (\neg (p \lor q)) \leftrightarrow ((\neg p) \land (\neg q)))\)

3. Show that logical and material equivalence coincide, that is, \(\models \varphi \leftrightarrow \psi\) iff \(\models (\varphi \leftrightarrow \psi)\).

4. Show that \(\models \varphi\) iff \((\varphi \land \neg \psi)\) is not satisfiable.

5. Prove that logical implication is reflexive, transitive, but not symmetric. Prove that logical equivalence is an equivalence relation.

6. Show that if \(\varphi\) and \(\psi\) are logically equivalent and \(\varphi\) is valid, then \(\psi\) is valid.

7. Which of the following two sentences implies the other?

   \((p \leftrightarrow (q \leftrightarrow r)),\)
   
   \(((p \land (q \land r)) \lor ((\neg p) \land ((\neg q) \land (\neg r))))\)

8. Show that \(\Sigma, \alpha \models \beta\) iff \(\Sigma \models (\alpha \rightarrow \beta)\) (Note: \(\Sigma, \alpha\) is a shorthand for \(\Sigma \cup \{\alpha\}\).)

9. Prove or refute the following statements (here \(\Sigma\) is a set of formulas):

   \(\bullet\) If either \(\Sigma \models \alpha\) or \(\Sigma \models \beta\), then \(\Sigma \models (\alpha \lor \beta)\).

   \(\bullet\) If \(\Sigma \models (\alpha \lor \beta)\), then either \(\Sigma \models \alpha\) or \(\Sigma \models \beta\).

10. A set \(\Sigma\) of formulas is \textit{independent} if no member \(\sigma\) of \(\Sigma\) is implied by \(\Sigma - \{\sigma\}\). Show that a finite set of formulas always has an equivalent independent subset, but this need not hold for an infinite set of formulas.
Substitutions:

The result of substituting a formula $\theta$ for a proposition $p$ in a formula $\varphi$, denoted $\varphi[p \mapsto \theta]$, is defined as follows.

- $p[p \mapsto \theta] = \theta$
- $q[p \mapsto \theta] = q$ for $q \neq p$.
- $(\neg \varphi)[p \mapsto \theta] = (\neg \varphi[p \mapsto \theta])$
- $(\varphi \circ \psi)[p \mapsto \theta] = (\varphi[p \mapsto \theta] \circ \psi[p \mapsto \theta])$ (here $\circ$ denoted an arbitrary binary connective)

Prove the following (use induction, but carefully!):

1. If $\varphi$ is valid, then $\varphi[p \mapsto \theta]$ is valid.
2. If $\alpha$ and $\beta$ are logically equivalent, then $\varphi[p \mapsto \alpha]$ and $\varphi[p \mapsto \beta]$ are logically equivalent.

Puzzle:

Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are (Box 1) “The gold is not here”, (Box 2) “The gold is not here”, and (Box 3) “The gold is in Box 2”. Only one message is true; the other two are false. Which box has the gold?

Formalize this puzzle and solve it using propositional logic.