Note:

1. The submitted assignment needs to be typeset in LaTeX.

2. **Teamwork:** You are expected to complete this assignment in teams. By signing your name on the assignment you are asserting that the work submitted was done collaboratively. At the very least, this entails:
   - discussing each problem and agreeing on a sketch of the solution,
   - reviewing the written solutions for technical correctness and typographical errors,
   - doing a joint codewalk over all written programs,
   - dividing labor equally (roughly), and
   - being able to explain every answer as if it is your own answer.

3. Note on the submission the total number of hours put in by each team member separately and by the team together.

**Note:** The pages below refer to the text from the book by Enderton (pdf posted).

1. Exercises 1, 3-6 on p. 78.
2. Exercises 1-6 on pp. 94-95.
3. Exercises 8-12 on p. 95.

**Notation:**

- $\Gamma; \alpha$ means $\Gamma \cup \{\alpha\}$
- $\models_A \varphi$ means $A \models \varphi$
- $\models_A \varphi(x)[a]$ means $A, [x \mapsto a] \models \varphi(x)$.
- $|A|$ refers to the domain of $A$.

4. Consider the following English sentences:
   - “There are some critics who admire only one another.”
   - “It is not the case that there are some numbers among which none is least”.

Can you formalize these sentences in first-order logic? How?

5. Show that the following formulas are valid, where in (b)-(i) $x$ is not free in $\beta$. Can the material implication in (a) be reversed?

(a) $\forall x (\alpha \rightarrow \beta) \rightarrow (\forall x \alpha \rightarrow \forall x \beta)$

(b) $\forall x (\alpha \land \beta) \leftrightarrow (\forall x \alpha \land \beta)$

(c) $\exists x (\alpha \land \beta) \leftrightarrow (\exists x \alpha \land \beta)$

(d) $\forall x (\alpha \lor \beta) \leftrightarrow (\forall x \alpha \lor \beta)$

(e) $\exists x (\alpha \lor \beta) \leftrightarrow (\exists x \alpha \lor \beta)$

(f) $\forall x (\alpha \rightarrow \beta) \leftrightarrow (\exists x \alpha \rightarrow \beta)$

(g) $\exists x (\alpha \rightarrow \beta) \leftrightarrow (\exists x \alpha \rightarrow \beta)$

(h) $\forall x (\beta \rightarrow \alpha) \leftrightarrow (\forall x \beta \rightarrow \forall x \alpha)$

(i) $\exists x (\beta \rightarrow \alpha) \leftrightarrow (\beta \rightarrow \exists x \alpha)$

6. Assume a relational vocabulary (i.e., no function symbols). For a sentence $\varphi$ of 1st-order logic with equality, let $\varphi'$ be the result of replacing every atomic formula $x = y$ in $\varphi$ by $E(x, y)$, where $E$ is a new binary predicate symbol, and then conjoining with the equivalence and congruence axioms for $E$. (The equivalence axioms says that $E$ is reflexive, symmetric and transitive. The congruence axioms says that if $P(a_1, \ldots, a_k)$ holds and $E(a_i, b_i)$ holds for $i = 1, \ldots, k$, then $P(b_1, \ldots, b_k)$ holds.) Show that $\varphi$ is satisfiable iff $\varphi'$ is satisfiable. (Recall that a sentence is satisfiable if it is satisfied by some structure.) (Hint: You can use equivalence classes as elements.)

7. An existential-conjunctive formula is a formula of the form $(\exists x_1) \ldots (\exists x_n) \wedge_{i=1}^{k} \alpha_i$, where each $\alpha_i$ is an atomic formula. What is the complexity (data and query complexity) of evaluating existential-conjunctive queries? (Focus on upper bounds.)

Consider a vocabulary of one binary relation symbol $R$. Let $\mathcal{A} = (D, R)$ be a structure with $D = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$. With each graph $G = (V, E)$ we associate a sentence $\varphi_G$ as follows. Let $V = \{v_1, \ldots, v_n\}$. Then $\varphi_G$ is

$$(\exists x_1) \ldots (\exists x_n) \bigwedge_{(v_i, v_j) \in E} R(x_i, x_j).$$

(Note that $\varphi_G$ is an existential-conjunctive formula.) Show that $\mathcal{A} \models \varphi_G$ iff $G$ is 3-colorable. What can you conclude from this about the complexity of evaluating existential-conjunctive queries? (Discuss upper and lower bounds.)

8. Drinker’s Principle: “In every group of people one can point to one person in the group such that if that person drinks then all the people in the group drink.”

Formulate this principle in first-order logic and prove its validity.