Read these instructions carefully, twice!

1. **You may use:** your textbook (Schöning), your notes, material on the course web page.

2. **You may not use:** other books, other web sites, material from other people or other classes.

3. **You may not discuss:** this exam with anyone else until the due date (others might still be taking the exam when you are already done).

4. You can take up to **five** hours to complete the exam. You can complete it in one or two seatings. You are responsible for keeping the time. You should write on your exam the precise times (start and end) during which you worked on the exam.

5. You should format your answers in LaTeX in a third seating, which is not included in the five-hour limit, but do note the start and end times of that seating.

6. **Due date:** Dec. 16, 2020, 11:59pm.

7. You can ask me questions of clarification about the exam, but I will not discuss with you possible approaches to solutions. You can ask me and the TA questions about the material taught in class and about the assignments.

8. All the questions in the exam carry equal weight.

9. Keep your answers clear, concise, to the point, and rigorous (try to avoid “clearly” and “easily”). It is not enough to write the idea of the solution, you have to show that it is correct. Please write very neatly. If you cannot get a complete solution, you may earn partial credit for partial solutions.

10. **Write your name on the exam in clear capital letters. Also, write and sign the honor pledge on your exam.**
1. A theory is a set of sentences closed under logical implication. If $S$ is a set of sentences, then $Th(S) = \{ \varphi : S \models \varphi \}$, i.e., $Th(S)$ is the closure of $S$ under logical implication. Thus, $T$ is a theory if $T = Th(T)$. A theory $T$ is finitely axiomatizable if there is a finite set $S$ of sentences such that $T = Th(S)$.

Let $T_0 \subset T_1 \subset T_2 \ldots$ be a strictly increasing sequence of theories.

(a) Use compactness to show that the union $T = \bigcup_{i=0}^{\infty} T_i$ is a theory and is satisfiable.
(b) Prove that $T$ is not finitely axiomatizable.

2. Let $A, B$ be two structures. We say that $A$ is a substructure of $B$ if the following hold:

- $D_A$ is a subset of $D_B$,
- For each $n$-place predicate symbol $P$ and each $n$-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $D_A$, we have that $\langle a_1, \ldots, a_n \rangle \in P_A$ iff $\langle a_1, \ldots, a_n \rangle \in P_B$.
- For each $n$-place function symbol $f$ and each $n$-tuple $\langle a_1, \ldots, a_n \rangle$ of elements of $D_A$, we have that $f_A(a_1, \ldots, a_n) = f_B(a_1, \ldots, a_n)$.

A first-order formula $\varphi$ is said to be universal if it is of the form $\forall x_1 \cdots \forall x_n \theta$, where $\theta$ is quantifier free. A sentence $\varphi$ is said to be preserved under substructures if it is true in a structure $A$ whenever it is true in a structure $B$ and $A$ is a substructure of $B$.

(a) Prove that every universal sentence is preserved under substructures. (Hint: Start with induction on quantifier-free formulas and then follow by induction on the number of universal quantifiers).
(b) Conclude that the sentence $\exists x P(x)$ is not logically equivalent to any universal sentence. Conclude also that the satisfiability problem for universal relational (i.e., no constant and function symbols) sentences is decidable.

3. A structure $A$ is a model of a sentence $\varphi$ if $A \models \varphi$. We use $\text{models}(\varphi)$ to denote the class of of models of $\varphi$.

(a) Show that if a sentence $\varphi$ has arbitrarily large finite models, then $\varphi$ also has an infinite model. (Hint: Use the Compactness Theorem.)
(b) Show that the first-order sentence

$$(\forall x)(\forall y)(\forall z)(\exists u)(E(x, u) \land ((E(x, y) \land E(y, z)) \rightarrow E(x, z)) \land (\neg E(x, x)))$$

has no finite models.
(c) Infer from item (a) that there is no first-order sentence $\varphi$ over the vocabulary $\Sigma$ consisting of the binary predicate symbols $E$ such that $\text{models}(\varphi)$ consists of precisely all finite structures over $\Sigma$. Infer from this that there is no first-order sentence $\varphi$ over the vocabulary $\Sigma$ consisting of the binary predicate symbols $E$ such that $\text{models}(\varphi)$ consists of precisely all infinite structures over $\Sigma$. Explain why this does not contradict the item (b).
4. Consider the formulas $\varphi_n(x, y)$ over the vocabulary $\Sigma$ consisting of the binary predicate symbols $E$, defined as follows, where $\approx$ is the equality predicate symbol:

- $\varphi_0(x, y)$ is $E(x, y)$
- $\varphi_n(x, y)$ is

$$(\exists z)(\forall u)(\forall v)((u \approx x \land v \approx z) \lor (u \approx z \land v \approx y)) \rightarrow (\forall x)(\forall y)((x \approx u \land y \approx v) \rightarrow \varphi_{n-1}(x, y)))$$

Please answer the following questions:

(a) View $\varphi_n(x, y)$ as a query over a database that consists of a directed graph. What query does $\varphi_n(x, y)$ express? (Hint: focus on the role of the universal quantifiers in $\varphi_n$. What do they accomplish?)

(b) Express $\varphi_n(x, y)$ as a logically equivalent formula $\psi_n(x, y)$ in prenex normal form using only existential quantifiers.

(c) How does the running time of $\text{Truth}(A, \alpha, \varphi_n)$ compare to the running time of $\text{Truth}(A, \alpha, \psi_n)$? What can you conclude from this about data complexity and query complexity? Can you think of a better way to evaluate the query expressed by $\varphi_n(x, y)$ (and $\psi_n(x, y)$)? What can you conclude from this?