Read the following directions carefully!

1. **You may use**: your textbook, your notebook, material on the course Web page, and your marked-up assignments.

2. **You may not use**: other books, solutions from previous years or other classes, other material on the Web.

3. **You may not discuss**: this exam with anyone else until the due date (others might still be taking the exam when you are already done).

4. **Please spend up to one hour** reading the directions and the exam carefully. If you have any questions, send them to the instructor as soon as possible.

5. You can take up to **six** hours to complete the exam. You can complete it in one or two seatings. You are responsible for keeping the time. You should write on your exam the precise times (start and end) during which you worked on the exam.

6. You should format your answers in LaTeX in a third seating, which is not included in the six-hours limit, but do note the start and end times of that seating.

7. **Due date**: Wednesday, Oct. 27, 2021, 11:59pm. Submit on Canvas.

8. You can ask (via email) questions of clarification about the exam, but I will not discuss with you possible approaches to solutions. You can also ask me or the TA questions about the material taught in class and about the assignments.

9. Every question in the exam carries 25 points. The exam will be graded on a curve.

10. Keep you answers clear, concise, to the point, and rigorous (avoid “clearly” and “easily”).

11. If you cannot get a complete solution, you may earn partial credit for partial solutions.

12. Do not forget to include your name and the honor pledge on your exam.
1. Suppose that we modify our definition of formulas by omitting all right parentheses. Thus, instead of

\[(A \land (\neg B)) \to (C \lor D)\]

we write

\[(A \land (\neg B) \to (C \lor D).\]

Describe this syntax formally and show that we still have unique readability.

2. The set of subformulas of a given formula is defined as follows:

- \(\text{subf}(p) = \{p\}\), when \(p\) is an atomic proposition
- \(\text{subf}((\neg \psi)) = \{(\neg \psi)\} \cup \text{subf}(\psi)\)
- \(\text{subf}((\psi \circ \theta)) = \{(\psi \circ \theta)\} \cup \text{subf}(\psi) \cup \text{subf}(\theta)\), for each binary connective \(\circ\)

Let \(\text{length}(\varphi)\) be the length of a formula \(\varphi\) as a string (each symbol, including atomic propositions, connectives, and parentheses, count as one letter). Find upper and lower bounds on the size of \(\text{subf}(\varphi)\) (i.e., the number of distinct subformulas of \(\varphi\)) in terms of \(\text{length}(\varphi)\). Try to make these bounds as tight as possible.

3. Show that for every propositional formula \(\varphi\) there is a propositional formula \(\psi\), such that

- \(\psi\) may include atomic propositions that do not occur in \(\varphi\) (hint: introduce a new proposition for each subformula),
- \(\psi\) can be constructed efficiently (i.e., in polynomial time) from \(\varphi\),
- \(\psi\) is in conjunctive normal form with at most 3 disjuncts in each conjunct, and
- \(\psi\) has the property that \(\varphi\) is satisfiable iff \(\psi\) is satisfiable.

(For simplicity assume that \(\varphi\) is constructed from the connectives \(\neg\) and \(\land\).) What can you conclude from this about NP-completeness of satisfiability?

4. The set 2-OCCUR is the set of propositional clausal formulas where every proposition occurs at most twice. For example, the formula \((p \lor q) \land (\neg p \lor r \lor s)\) is in 2-OCCUR, while \((p \lor q) \land (\neg p \lor r) \land (p \lor \neg s)\) is not in 2-OCCUR. Show that the satisfiability problem for formulas in 2-OCCUR is in PTIME (hint: use resolution).

5. Prove the Interpolation Theorem for propositional logic: if \(\varphi \models \psi\) for formulas \(\varphi\) and \(\psi\), then there is a formula \(\theta\) (possibly 0 or 1) such that

- \(\text{AP}(\theta) \subseteq \text{AP}(\varphi) \cap \text{AP}(\psi)\),
- \(\varphi \models \theta\),
- \(\theta \models \psi\)

The formula \(\theta\) is called an interpolant; provide an upper bound on its length (in terms of the lengths of \(\varphi\) and \(\psi\)).