Note: The pages below refer to the text from the book by Enderton.

1. Exercises 1-6 on p. 78.

1. Translate into this language the English sentences listed below. If the English sentence is ambiguous, you will need more than one translation.

(a) Zero is less than any number.
\[(\forall n)(N(n)) \rightarrow (0 < n)\]

(b) If any number is interesting, then zero is interesting.
Here are two interpretations of this sentence:
\[(\exists n)(N(n) \land I(n)) \rightarrow I(0)\]
also, we could interpret it as \[(\forall n)(N(n) \land I(n) \rightarrow I(0))\]

(c) No number is less than zero.
\[\neg(\exists n)(N(n) \land (n < 0))\]
or, equivalently, \[(\forall n)\neg((N(n) \land (n < 0))\]

(d) Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting.
\[(\forall n)(N(n) \land \neg I(n) \land ((\forall m)((N(m)) \land (m < n)) \rightarrow (I(m)))) \rightarrow I(n)\] (which is a contradiction!)

(e) There is no number such that all numbers are less than it.
\[\neg(\exists n)(N(n)) \land (\forall m)(N(m) \rightarrow (m < n))\]

(f) There is no number such that no number is less than it.
\[\neg(\exists n)(N(n)) \land (\neg(\exists m)(N(m) \land (m < n))))\]
2. Neither a nor b is a member of every set.

\[ \neg(\forall s(a \in S)) \land \neg(\forall s(b \in S)) \]

3. If horses are animals, then heads of horses are heads of animals. (E is a horse; A is an animal; h() designates a head.)

\[ (\forall x(E(x) \rightarrow A(x))) \rightarrow (\forall x((E(x) \rightarrow (\exists y(A(y) \land (h(x) \approx h(y))))) \]

4. P is a person; T is a time, F(x,y) means that you can fool x at time y.

(a) You can fool some of the people all of the time.

\[ (\exists p)(\forall t)(P(p)) \land (T(t) \rightarrow (F(p,t))) \]

(b) You can fool all of the people some of the time.

\[ (\exists t)(\forall p)(P(p)) \land (T(t) \rightarrow (F(p,t))) \]

(c) You can't fool all of the people all of the time.

\[ \neg((\forall p,t)(P(p) \land (T(t) \rightarrow (F(p,t))))) \]

5. J is a job; a designates Adams; D(x,y) means that x can do y right.

(a) Adams can't do every job right.

\[ \neg(\forall j)(J(j) \rightarrow (D(a,j))) \]

(b) Adams can't do any job right.

\[ (\forall j)(J(j) \rightarrow (\neg D(a,j))) \]

6. Nobody likes everybody. (L(x,y) means x likes y.)

\[ \neg((\exists x)(\forall y)(L(x,y))) \]

2. Exercises 1-6 on pp. 94-95.

1. Show that

(a) \( \Gamma; \alpha \models \varphi \) iff \( \Gamma \models (\alpha \rightarrow \varphi) \)

\[ \Rightarrow: \]

To show this holds, assume \( \Gamma; \alpha \models \varphi \). We show that for every domain \( B \) and assignment \( \beta \), if \( B; \beta \models \Gamma \), then \( B; \beta \models (\alpha \rightarrow \varphi) \), which is the definition of \( \Gamma \models (\alpha \rightarrow \varphi) \). Consider an arbitrary domain \( B \) and assignment \( \beta \). The premise, that \( \Gamma; \alpha \models \varphi \), implies that if \( B; \beta \models \Gamma; \alpha \) then \( B, \beta \models \varphi \). By definition, \( B; \beta \models \Gamma; \alpha \) when \( B; \beta \models \Gamma \) and \( B; \beta \models \alpha \). We thus consider
three cases: that \( B; \beta \models \Gamma; \alpha \), that \( B; \beta \models \Gamma \) but \( B; \beta \not\models \alpha \), and that \( B; \beta \not\models \Gamma \). In the first case, our premise entails that \( B; \beta \models \varphi \), and thus by definition of \( \rightarrow \) we have \( B; \beta \models (\alpha \rightarrow \varphi) \).

In the second case, where \( B; \beta \models \Gamma \) but \( B; \beta \not\models \alpha \), we cannot draw any conclusions from our premise. However, since \( B; \beta \not\models \alpha \), by definition of \( \rightarrow \) we have that \( B; \beta \not\models (\alpha \rightarrow \varphi) \). In the third case, where that \( B; \beta \not\models \Gamma \), our conclusion holds trivially.

\[ \iff \]
To show the reverse direction holds, assume \( \Gamma \models (\alpha \rightarrow \varphi) \). We show that for every domain \( B \) and assignment \( \beta \), if \( B; \beta \models \Gamma ; \alpha \), then \( B; \beta \models \varphi \). We consider the same two cases: either both \( B; \beta \models \Gamma \) and \( B; \beta \models \alpha \), or not. In the first case, since \( B; \beta \models \Gamma \), our premise implies \( B; \beta \models (\alpha \rightarrow \varphi) \). Since \( B; \beta \models \alpha \), this holds only when \( B; \beta \models \varphi \). In the second case, one of that \( B; \beta \models \Gamma \) and \( B; \beta \models \alpha \) fails to hold, which implies \( B; \beta \not\models \Gamma; \alpha \), and our conclusion holds.

(b) \( \varphi \models \psi \iff (\varphi \leftrightarrow \psi) \)

Using the result from above, we have,

\[
\varphi \models \psi \iff \varphi \models \psi \text{ and } \psi \models \psi \\
\iff \models (\varphi \rightarrow \psi) \text{ and } \models (\psi \rightarrow \varphi) \\
\iff \models (\varphi \leftrightarrow \psi)
\]

2. Show that no one of the following sentences is logically implied by the other two. (This is done by giving a structure in which the sentence in question is false, while the other two are true.)

(a) \( \forall x \forall y \forall z (Pxy \rightarrow (Pyz \rightarrow Pxz)) \)

\[ D = \mathbb{R}, P(x, y) \text{ is true iff } (y = \sqrt{x}) \]

(b) \( \forall x \forall y (Pxy \rightarrow (Pyx \rightarrow x \approx y)) \)

\[ D = G\{V, E\} \text{ such that } \text{G is a clique}, P(x, y) \text{ is true iff there is an edge connecting } x \text{ to } y \]

(c) \( \forall x \exists y Pxy \rightarrow \exists y \forall x Pxy \)

\[ D = \mathbb{R}, P(x, y) \text{ is true iff } (x \leq y) \]

3. Show that \( \{\forall x (\alpha \rightarrow \beta), \forall x \alpha \} \models \forall x \beta \).

\[
\{\forall x (\alpha \rightarrow \beta), \forall x \alpha \} \models \forall x \beta \iff \models (\forall x (\alpha \rightarrow \beta) \land \forall x \alpha) \rightarrow \forall x \beta) \\
\iff \models (\forall x (\neg \alpha \lor \beta) \land \forall x \alpha) \rightarrow \forall x \beta) \\
\iff \models ((\forall x \text{false} \lor \beta)) \rightarrow \forall x \beta) \\
\iff \models ((\forall x \beta) \rightarrow \forall x \beta),
\]

which is trivially true.

4. Show that if \( x \) does not occur free in \( \alpha \), then \( \alpha \models \forall x \alpha \).
We show that for every domain \( B \) and assignment \( \beta \), if \( B; \beta \models \alpha \), then \( B; \beta \models \forall x \alpha \). Consider a \( B; \beta \). If \( B; \beta \not\models \alpha \), our conclusion holds trivially. Thus let \( B; \beta \) be such that \( B; \beta \models \alpha \). Since \( x \) is not in \( \text{FVars}(\alpha) \), for every \( d \) it holds that \( \beta|_{\text{FVars}(\alpha)} = \beta[x \mapsto d]|_{\text{FVars}(\alpha)} \). By the Relevance Lemma of FOL, we can thus conclude that for every \( d \), \( B; \beta[x \mapsto d] \models \alpha \). Since \( B; \beta \models \alpha \) is our premise, we have that for every \( d \), \( B; \beta[x \mapsto d] \models \alpha \): the definition of \( B; \beta \models \forall x \alpha \).

5. Show that the formula \( \varphi = (x \approx y) \rightarrow (Pzfx \rightarrow Pzf y) \) (where \( f \) is a one-place function symbol and \( P \) is a two-place predicate symbol) is valid.

We’ll prove a stronger claim \( \varphi = ((x \approx y) \rightarrow (P(z, f(x)) \iff (P(z, f(y)))) \)

**Proof.**

Let’s select a structure \( A \).

1) \( A, \alpha \not\models (x \approx y) \rightarrow A, \alpha \models \varphi \) by the definition of \( \rightarrow \).

2) \( A, \alpha \models (x \approx y) \rightarrow \alpha(x) = \alpha(y) \). Thus, \( A, \alpha \models P(z, f(x)) \) iff \( P^A(\alpha(z), f^A(\alpha(y))) \) iff \( P^A(\alpha(z), f^A(\alpha(y))) \) iff \( A, \alpha \models P(z, f(y)) \). Therefore, \( A, \alpha \models (P(z, f(x)) \iff (P(z, f(y)))) \rightarrow A, \alpha \models \varphi \).

Combining the two cases, we conclude that \( \varphi \) is valid.

6. Show that a formula \( \theta \) is valid iff \( \forall x \theta \) is valid.

Let’s select a structure \( A \).

\( \Rightarrow \) Let’s select an \( \alpha \). \( \models \theta \rightarrow \) for all \( a \in D \), \( A, \alpha[x \mapsto a] \models \theta \) iff \( A, \alpha \models (\forall x \theta) \). Because it is true for all \( \alpha \), \( (\forall x \theta) \) is valid.

\( \Leftarrow \) Suppose there exists \( \alpha \), \( A, \alpha \not\models \theta \rightarrow \) exists \( \alpha \), \( A, \alpha \models (\neg \theta) \rightarrow \) exists \( a \in D \) \( (a = \alpha(x)) \), \( A, \alpha[x \mapsto a] \models (\neg \theta) \rightarrow A, \alpha \models (\exists x (\neg \theta)) \).

On the contrary, for the same \( \alpha \), \( A, \alpha \models (\forall x \theta) \iff A, \alpha \models (\neg (\exists x (\neg \theta))) \). Because \( \alpha \) cannot satisfy a formula and the same negated formula at the same time, the obtained contradiction concludes the proof.

3. Exercises 8-12 on p. 95.

8. Assume that \( \Sigma \) is a set of sentences such that for any sentence \( \tau \), either \( \Sigma \models \tau \) or \( \Sigma \models \neg \tau \). Assume that \( \mathcal{U} \) is a model of \( \Sigma \). Show that for any sentence \( \tau \), \( \models_{\mathcal{U}} \tau \) iff \( \Sigma \models \tau \).

1) We will prove first that if \( \Sigma \models \tau \) then \( \models_{\mathcal{U}} \tau \). \( \models_{\mathcal{U}} \Sigma \) and \( \Sigma \models \tau \) implies \( \models_{\mathcal{U}} \tau \).

2) Assume \( \models_{\mathcal{U}} \tau \). We will prove by contradiction that \( \Sigma \models \tau \). Assume that \( \Sigma \not\models \tau \). Then, by hypothesis, \( \Sigma \models \neg \tau \). By point 1, it follows that \( \models_{\mathcal{U}} \neg \tau \). But then \( \models_{\mathcal{U}} \neg \tau \) and \( \models_{\mathcal{U}} \tau \), contradiction. Then \( \Sigma \models \tau \).

9. Assume that the language has equality and a two-place predicate symbol \( P \). For each of the following conditions, find a sentence \( \sigma \) such that the structure \( \mathcal{U}(= ([\mathcal{U}], P^{\mathcal{U}})) \) is a model of \( \sigma \) iff the condition is met.
(a) \(|\mathcal{U}|\) has exactly two members.

\[ \exists x \exists y \forall z (\neg (x \approx y) \land (z \approx x \lor z \approx y)) \]

Exists two different elements but choose any element must me the same of one of the two.

(b) \(P^U\) is a function from \(|\mathcal{U}|\) into \(|\mathcal{U}|\).

\[ \forall x \exists y \forall z (P(x, y) \land (P(x, z) \rightarrow y \approx z)) \]

For \(P\) to be \(x \rightarrow y\), there is a \(y\) for every \(x\), and whenever \(P(x) = z\), \(z = y\).

(c) \(P^U\) is a permutation of \(|\mathcal{U}|\); i.e., \(P^U\) is a one-to-one function with domain and range equal to \(|\mathcal{U}|\).

\[ (\forall x \exists y \forall z (P(x, y) \land (P(x, z) \rightarrow y \approx z))) \land (\forall y \exists x \forall z (P(x, y) \land (P(z, y) \rightarrow x \approx z))) \]

\(P\) is both a function from \(x\) to \(y\) and from \(y\) to \(x\).

10. **Show that** \(\models_{\mathcal{U}} \forall v_2 \ Qv_1 v_2[\langle c^U \rangle] \iff \models_{\mathcal{U}} \forall v_2 \ Qc v_2\). **Here \(Q\) is a two-place predicate symbol and \(c\) is a constant symbol.**

\[ \Rightarrow: \]

Take \(A\) so that \(A, [v_1 \mapsto c^A] \models \forall v_2 Q(v_1, v_2)\). This is equivalent to for all \(d \in |A|\), \(A, [v_1 \mapsto c^A, v_2 \mapsto d] \models Q(v_1, v_2)\). i.e. For all \(d \in |A|\), \(<c^A, d> \in Q^A\). So \(A \models \forall v_2 Q(c, v_2)\).

\[ \Leftarrow: \]

Take \(A\) so that \(A \models \forall v_2 Q(c, v_2)\). For all \(d \in |A|\), \(A, [v_2 \mapsto d] \models Q(c, v_2)\). So \(<c^A, d> \in Q^A\). So \(A, [v_1 \mapsto c^A] \models \forall v_2 Q(v_1, v_2)\)

11. **For each of the following relations, give a formula which defines it in \((\mathbb{N}, +, \cdot)\). (The language is assumed to have equality and the parameters \(\forall, +, \text{and} \cdot\).)**

(a) \(\{0\}\).

\[ \text{zero}(z) = (\forall n)(n + z \approx n) \]

\[ \text{zero}(z) = (\forall n)((n \cdot z) \approx z) \]

(b) \(\{1\}\).

\[ \text{one}(o) = (\forall n)((n + o) \approx n) \]

(c) \(\{\langle m, n \rangle : n \text{ is the successor of } m \text{ in } \mathbb{N}\}\).

\[ \text{successor}(m, n) = (\exists x)(\text{one}(x) \land (n \approx (m + x))) \]

(d) \(\{\langle m, n \rangle : m < n \text{ in } \mathbb{N}\}\).

\[ \text{lessthan}(m, n) = (\exists k)(\neg \text{zero}(k) \land (n \approx (m + k))) \]

5
12. Let $\mathbb{R}$ be the structure $(\mathbb{R}, +, \cdot)$. (The language is assumed to have equality and the parameters $\forall$, $+$, and $\cdot$. $\mathbb{R}$ is the structure whose universe is the set $\mathbb{R}$ of real numbers and such that $+$ and $\cdot$ are the usual addition and multiplication operations.)

(a) Give a formula which defines in $\mathbb{R}$ the set $[0, \infty)$.

$$P(s) = (\exists r)(s \approx (r \cdot r))$$

(b) Give a formula which defines in $\mathbb{R}$ the set $\{2\}$.

$$1(x) = (\forall r)(r \cdot x \approx r)$$

$$2(x) = (\exists y)(1(y) \land (x \approx (y + y)))$$

(c) Show that any finite union of intervals, the endpoints of which are algebraic, is definable in $\mathbb{R}$. (The converse is also true; these are the only definable sets in the structure. But we will not prove this fact.)

A note on notation: Given a formula $\varphi$ with a single free variables $x$, we will sometimes denote it $\varphi(x)$ to make the free variable explicit.

We first note that the natural linear order ($\leq$) and the natural strict linear order ($<$) on $\mathbb{R}$ can be defined as follows:

- $(x \leq y) = \exists z(y \approx (x + z \cdot z))$
- $(x < y) = (x \leq y) \land (x \neq y)$

For each integer $m$, we construct a formula $\varphi_m(x)$ that defines $m$. We first define 0 and 1:

- $\varphi_0(x) = \forall y(x + y \approx y)$
- $\varphi_1(x) = \forall y(x \cdot y \approx y)$

Then, for natural number $n > 1$, $\varphi_n$ is inductively defined as

$$\varphi_n(x) = \exists y\exists z(x \approx y + z) \land \varphi_{n-1}(y) \land \varphi_1(z)$$

and the negative integer $-n$ is defined by the formula

$$\varphi_{-n}(x) = \exists y\exists z(x + y \approx z) \land \varphi_n(y) \land \varphi_0(z)$$

We note that we can now treat each integer $m$ as a constant while defining new formulas. Our justification for this simplification is as follows: Given a formula $\varphi$ with occurences of $m$ as a constant term, we can obtain a formula $\varphi'$ by replacing each occurrence of $m$ with a fresh variable $x_m$. Then $\varphi'' = \exists x_m(\varphi_m(x_m) \land \varphi')$ is logically equivalent to $\varphi$ and does not contain any occurence of $m$. Thus using integer constants in our formulas gives us no additional power and is merely a notational simplification.

We next show that each algebraic number can be defined in $\mathbb{R}$. By definition, each algebraic number is the root of a polynomial with integer coefficients. Given a polynomial $p(x)$, we inductively define the corresponding term $t_{p(x)}$ as follows:

- For $c \in \mathbb{Z}$, $t_c = c$
Then the roots of the polynomial \( p(x) \) can be defined by the formula \( \varphi_{p(x)=0} \) as follows:

\[
\varphi_{p(x)=0} = (t_{p(x)} \equiv 0)
\]

In general, \( p(x) \) can have multiple roots, so we need to distinguish between the roots and obtain the one we want. Let \( \alpha \) be an algebraic number such that \( \alpha \) is a root of \( p(x) \). Since \( p(x) \) has a finite number \( n \) of distinct roots, we can order these roots using \( < \). Then there exists \( k \leq n \) such that \( \alpha \) is the \( k \)th smallest root. We can then define \( \alpha \) as follows:

\[
\varphi_{\alpha}(x) = \exists x_1 \cdots \exists x_n ((x \equiv x_k) \bigwedge_{i=1}^{n} \varphi_{p(x_i)=0} \bigwedge_{i<k} (x_i < x_k) \bigwedge_{j>k} (x_j > x_k) \bigwedge_{i \neq j} (x_i \neq x_j))
\]

Therefore every algebraic number can be defined. Let \( \alpha \) and \( \beta \) be algebraic numbers with \( \alpha \leq \beta \). Then the following formula defines the interval \([\alpha, \beta]\) (here we only give the case for the closed interval; the other types of intervals can be defined similarly):

\[
\varphi_{[\alpha, \beta]}(x) = \exists y \exists z (\varphi_{\alpha}(y) \land \varphi_{\beta}(z) \land (y \leq x) \land (x \leq z))
\]

Finally, a finite union of intervals can be defined by simply taking the conjunction of the formulas for each interval in the collection. Thus the union of the intervals \([\alpha_1, \beta_1], \ldots , [\alpha_k, \beta_k]\) can be defined by the formula:

\[
\bigvee_{i=1}^{k} \varphi_{[\alpha_i, \beta_i]}(x)
\]

4. Exercise 17(a) on p. 96: Moved to HW6

5. Consider the following English sentences. Can you formalize these sentences in first-order logic? How?

- “There are some critics who admire only one another.”

  We can interpret this statement as defining a clique on a graph, where there is an edge between two critics iff they admire one another. Since FOL is not powerful enough to express cliques, this sentence cannot be expressed either.

- “It is not the case that there are some numbers among which none is least”.

  Notice that here again there is a second-order quantification in disguise. We would like to be able to quantify over sets of numbers, but this is not allowed in FOL.

6. Show that the following formulas are valid, where in (b)-(i) \( x \) is not free in \( \beta \). Can the material implication in (a) be reversed?

  We use the following lemma in our derivations below.
Lemma: $x \notin FVars(\beta) \to \beta \models \forall x \beta$.

Proof: By the relevance lemma, $\alpha \models_{FVars(\beta)} \alpha[x \mapsto d] \models_{FVars(\beta)} \beta$ for all $d \in D$ and so $A, \alpha \models [x \mapsto d] \models \beta$ if $A, \alpha \models \beta$ for all $d \in D$. Then,

$$models(\forall x \beta) = \{ A, \alpha \models \beta \}$$

(a) $\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta)$

$\forall x(\psi \to \beta)$ iff $A, \alpha \models \forall x(\psi \to \beta)$ iff, for all $a \in D$, $A, \alpha \models [x \mapsto a] (\nsim \psi)$ or $A, \alpha \models [x \mapsto a] \models \beta \Rightarrow$ exists $a \in D$, $A, \alpha \models [x \mapsto a] (\nsim \psi)$ or for all $a \in D$, $A, \alpha \models [x \mapsto a] \models \beta$ iff $(\exists x(\nsim \psi) \lor (\forall x \beta))$ iff $(\forall x \nsim \psi \rightarrow \forall x \beta)$

In this part the material implication cannot be reversed because in the step “for all $a \in D$, $A, \alpha \models [x \mapsto a] \models (\nsim \psi)$ or $A, \alpha \models [x \mapsto a] \models \beta \Rightarrow$ exists $a \in D$, $A, \alpha \models [x \mapsto a] (\nsim \psi)$ or for all $a \in D$, $A, \alpha \models [x \mapsto a] \models \beta”$, the implication holds only in that one direction.

(b) $\forall x(\alpha \land \beta) \leftrightarrow (\forall x \alpha \land \beta)$

$$\forall x(\alpha \land \beta) \leftrightarrow (\forall x \alpha \land \forall x \beta) \text{ by distributivity of } \forall$$

$$\leftrightarrow (\forall x \alpha \land \beta) \text{ because } x \text{ is not free in } \beta$$

(c) $\exists x(\alpha \land \beta) \leftrightarrow (\exists x \alpha \land \beta)$

$$\exists x(\alpha \land \beta) \leftrightarrow (\exists x \alpha \land \exists x \beta) \text{ by the lemma}$$

$$\leftrightarrow (\exists x \alpha \land \beta) \text{ because } x \text{ is not free in } \beta$$

(d) $\forall x(\alpha \lor \beta) \leftrightarrow (\forall x \alpha \lor \beta)$

$$\forall x(\alpha \lor \beta) \leftrightarrow (\forall x \alpha \lor \forall x \beta) \text{ by the lemma}$$

$$\leftrightarrow (\forall x \alpha \lor \beta) \text{ because } x \text{ is not free in } \beta$$

(e) $\exists x(\alpha \lor \beta) \leftrightarrow (\exists x \alpha \lor \beta)$

$$\exists x(\alpha \lor \beta) \leftrightarrow (\exists x \alpha \lor \exists x \beta) \text{ by distributivity of } \exists$$

$$\leftrightarrow (\exists x \alpha \lor \beta) \text{ because } x \text{ is not free in } \beta$$

(f) $\forall x(\alpha \to \beta) \leftrightarrow (\exists x \to \beta)$

$$\forall x(\alpha \to \beta) \leftrightarrow \forall x(\nsim \alpha \lor \beta) \text{ by definition of } \to$$

$$\leftrightarrow (\forall x \nsim \alpha \lor \forall x \beta) \text{ by the lemma}$$

$$\leftrightarrow (\nsim \exists x \alpha \lor \forall x \beta) \text{ by complementation of } \forall$$

$$\leftrightarrow (\exists x \to \beta) \text{ because } x \text{ is not free in } \beta$$

$$\leftrightarrow (\exists x \to \beta) \text{ by definition of } \to$$
(g) \(\exists x (\alpha \rightarrow \beta) \iff (\forall x \alpha \rightarrow \beta)\)

\(\exists x (\alpha \rightarrow \beta) \iff \exists x (\neg \alpha \vee \beta)\) by definition of \(\rightarrow\)
\(\iff (\exists x \neg \alpha \vee \exists x \beta)\) by distributivity of \(\exists\)
\(\iff (\neg \forall x \alpha \vee \exists x \beta)\) by complementation of \(\exists\)
\(\iff (\neg \forall x \alpha \vee \beta)\) because \(x\) is not free in \(\beta\)
\(\iff (\forall x \alpha \rightarrow \beta)\) by definition of \(\rightarrow\)

(h) \(\forall x (\beta \rightarrow \alpha) \iff (\beta \rightarrow \forall x \alpha)\)

\(\forall x (\beta \rightarrow \alpha) \iff \forall x (\neg \beta \vee \alpha)\) by definition of \(\rightarrow\)
\(\iff (\forall x \neg \beta \vee \forall x \alpha)\) by the lemma
\(\iff (\neg \exists x \beta \vee \forall x \alpha)\) by complementation of \(\forall\)
\(\iff (\neg \beta \vee \forall x \alpha)\) because \(x\) is not free in \(\beta\)
\(\iff (\beta \rightarrow \forall x \alpha)\) by definition of \(\rightarrow\)

(i) \(\exists x (\beta \rightarrow \alpha) \iff (\beta \rightarrow \exists x \alpha)\)

\(\exists x (\beta \rightarrow \alpha) \iff \exists x (\neg \beta \vee \alpha)\) by definition of \(\rightarrow\)
\(\iff (\exists x \neg \beta \vee \exists x \alpha)\) by distributivity of \(\exists\)
\(\iff (\neg \forall x \beta \vee \exists x \alpha)\) by complementation of \(\exists\)
\(\iff (\neg \beta \vee \exists x \alpha)\) because \(x\) is not free in \(\beta\)
\(\iff (\beta \rightarrow \exists x \alpha)\) by definition of \(\rightarrow\)

7. Assume a relational vocabulary (i.e., no function symbols). For a sentence \(\varphi\) of 1st-order logic with equality, let \(\varphi'\) be the result of replacing every atomic formula \(x = y\) in \(\varphi\) by \(E(x, y)\), where \(E\) is a new binary predicate symbol, and then conjoining with the equivalence and congruence axioms for \(E\). (The equivalence axioms says that \(E\) is reflexive, symmetric and transitive. The congruence axioms says that if \(P(a_1, \ldots, a_k)\) holds and \(E(a_i, b_i)\) holds for \(i = 1, \ldots, k\), then \(P(b_1, \ldots, b_k)\) holds.) Show that \(\varphi\) is satisfiable iff \(\varphi'\) is satisfiable. (Recall that a sentence is satisfiable if it is satisfied by some structure.) (Hint: You can use equivalence classes as elements.)

**Notation:** Given a formula \(\psi\) over a vocabulary that does not contain \(E\), we define \(\psi^E\) to be the formula obtained from \(\psi\) by replacing every atomic formula \((x \approx y)\) by \(E(x, y)\). We also define \(\sigma\) to be the sentence obtained by conjoining the equivalence and congruence axioms for equality. We note that in this notation, we have \(\varphi' = \varphi^E \land \sigma^E = (\varphi \land \alpha)^E\).

The sentence \(\sigma^E\) is given by the conjunction of the following sentences

\[
\begin{align*}
\varphi_{\text{reflexive}} & = \forall x E(x, x) \\
\varphi_{\text{symmetric}} & = \forall x \forall y (E(x, y) \rightarrow E(y, x)) \\
\varphi_{\text{transitive}} & = \forall x \forall y \forall z (E(x, y) \land E(y, z) \rightarrow E(x, z)) \\
\varphi_{\text{congruence}} & = \forall x_1 \ldots \forall x_k \forall y_1 \ldots \forall y_k ((P(x_1, \ldots, x_k) \land E(x_1, y_1) \land \cdots \land E(x_k, y_k)) \rightarrow P(y_1, \ldots, y_k))
\end{align*}
\]
where a congruence axiom is needed for each predicate symbol $P$ in the vocabulary.

**Only If:**

Let $\varphi$ be satisfiable. Then there exists a structure $A$ such that $A \models \varphi$. Let $D$ be the domain of $A$. Let $\psi = \varphi \land \sigma$. Since every structure with equality is a model of $\sigma$, therefore $A$ is also a model of $\psi$. Consider the structure $B$ that consists of the structure $A$ extended with the binary relation $E^B$ defined as follows: $E^B = \{(a, a) : a \in D\}$. Thus, $E^B$ is the equality relation on $D$. Since $A$ is a model of $\psi$ and the symbol $E$ does not occur in $\psi$, therefore $B$ is also a model of $\psi$. Finally, note that $\varphi'$ is obtained from $\psi$ by replacing each atomic formula $(x \approx y)$ by $E(x, y)$. Since $\approx$ and $E$ have the same interpretation in $B$, therefore $B$ is also a model of $\varphi'$. Thus, $\varphi'$ is satisfiable.

**If:**

Now suppose $\varphi'$ is satisfiable. Let $B$ be a structure such that $B \models \varphi'$. Then $B \models \varphi^E \land \sigma^E$. Since $B$ is a model of $\sigma^E$, therefore $E^B$ is a congruence relation for the structure $B$.

For an element $a \in D^B$, we use $[a]$ to denote the equivalence class of $a$ with respect to $E^B$. Formally, $[a] = \{a' \in D^B : (a, a') \in E^B\}$. We use $B$ to define a structure $A$ as follows: $D^A = \{[a] : a \in D^B\}$, and for each $k$-ary predicate $P^B$ in $B$, define a $k$-ary predicate $P^A$ in $A$ such that $([a_1], \ldots, [a_k]) \in P^A$ iff $(a_1, \ldots, a_k) \in P^B$. Note that each predicate $P^A$ is well defined because $E^B$ is a congruence. Also note that the domain of $A$ is the set of equivalence classes of $E^B$. Given a binding $\beta : \text{Var} \to D^B$, we define the corresponding binding $\alpha : \text{Var} \to D^A$ as follows: for all $x \in \text{Var}$, $\alpha(x) = [\beta(x)]$.

Let $\theta$ be an arbitrary formula that does not contain the symbol $E$. We prove, by structural induction, that $A, \alpha \models \theta$ iff $B, \beta \models \theta^E$. We note that since the vocabulary contains only relational symbols, the only terms are variables.

**Basis:** There are two possibilities for the smallest formula:

- $\theta = P(x_1, \ldots, x_k)$. Then $\theta^E = \theta$.

  $A, \alpha \models P(x_1, \ldots, x_k)$
  
  iff $\langle \alpha(x_1), \ldots, \alpha(x_k) \rangle \in P^A$ (Defn. of $\models$)
  
  iff $([\beta(x_1)], \ldots, [\beta(x_k)]) \in P^A$ (Defn. of $\alpha$)
  
  iff $\langle \beta(x_1), \ldots, \beta(x_k) \rangle \in P^B$ (Defn. of $P^A$)
  
  iff $B, \beta \models P(x_1, \ldots, x_k)$ (Defn. of $\models$)

- $\theta = (x \approx y)$. Then $\theta^E = E(x, y)$.

  $A, \alpha \models (x \approx y)$
  
  iff $\alpha(x) = \alpha(y)$ (Defn. of $\models$ and $\approx$)
  
  iff $[\beta(x)] = [\beta(y)]$ (Defn. of $\alpha$)
  
  iff $\langle \beta(x), \beta(y) \rangle \in E^B$ (Defn. of equivalence class)
  
  iff $B, \beta \models E(x, y)$ (Defn. of $\models$)

**Induction Step:** We need to consider the following three possibilities:
• $\theta = (\theta_1 \land \theta_2)$. Then,

\[
A, \alpha \models (\theta_1 \land \theta_2) \\
\text{iff} \ A, \alpha \models \theta_1 \text{ and } A, \alpha \models \theta_2 \quad \text{(Semantics of } \land) \\
\text{iff} \ B, \beta \models \theta^E_1 \text{ and } B, \beta \models \theta^E_2 \quad \text{(Induction Hypothesis)} \\
\text{iff} \ B, \beta \models (\theta^E_1 \land \theta^E_2) \quad \text{(Semantics of } \land)
\]

• $\theta = (\neg \theta')$. Then,

\[
A, \alpha \models (\neg \theta') \\
\text{iff} \ A, \alpha \nvDash \theta' \quad \text{(Semantics of } \neg) \\
\text{iff} \ B, \beta \nvDash \theta'^E \quad \text{(Induction Hypothesis)} \\
\text{iff} \ B, \beta \models (\neg \theta'^E) \quad \text{(Semantics of } \neg) \\
\text{iff} \ B, \beta \models (\neg \theta')^E
\]

• $\theta = \forall x \theta'$. Then,

\[
A, \alpha \models \forall x \theta' \\
\text{iff} \ \text{for all } [a] \in D^A, A, \alpha[x \mapsto [a]] \models \theta' \quad \text{(Semantics of } \forall) \\
\text{iff} \ \text{for all } a \in D^B, A, \alpha[x \mapsto [a]] \models \theta' \quad \text{(I.H. and Defn. of } \alpha) \\
\text{iff} \ B, \beta \models \forall x(\theta'^E) \quad \text{(Semantics of } \forall) \\
\text{iff} \ B, \beta \models (\forall x \theta')^E
\]

Since $B$ satisfies the sentence $\varphi' = \varphi^E \land \sigma^E$, it also satisfies the sentence $\varphi^E$. Then $B, \beta \models \varphi^E$ for any binding $\beta$. Then, by the result above, $A, \alpha \models \varphi$. Since $\varphi$ is also a sentence, by the relevance lemma, $A \models \varphi$. Thus, $\varphi$ is satisfiable.

8. **Problem Complexity of existential-conjunctive queries of a formula $\varphi$.**

Suppose the domain $D$ is finite and of cardinality $m$. We can evaluate any given assignment in time $\sum |\alpha_i|$. Thus an upper bound for evaluation is $O((\sum |\alpha_i|) \cdot m^n)$, and this problem is in NP: guess such an assignment.

That the problem is in fact NP-complete can be derived from the following theorem:

**Theorem 1.** *3-colorability can be expressed as existential-conjunctive queries.*

**Proof.** For a graph $G = (V, E)$ where $|V| = n$, take $\varphi_G = \exists x_1 \ldots \exists x_n \bigwedge_{(v_i, v_j) \in E}(R(x_i, x_j))$.

• Assume $\varphi_G$ is satisfiable. Then we know there are some assignment $a$ such that $A, a \models \bigwedge_{(v_i, v_j) \in E}(R(x_i, x_j))$. For each vertex $v_i$, we assign it color $x_i$. The fact that $A, a \models R(x_i, x_j)$ for all edges ensures there are no edge with the same color.
• Assume the $G$ is 3-colorable. Take a 3-coloring and construct the assignment $a$. $A, a \models R(x_i, x_j)$ since $x_i \neq x_j$ since it is a 3-coloring. So $A \models \varphi_G$.

9. “In every group of people one can point to one person in the group such that if that person drinks then all the people in the group drink”. Formulate this principle in first-order logic and prove its validity.

We model different groups of people by using different values of the domain in our structures. The structure gives us a group of people, so the domain of the structure will be $D = G = \{\text{a set of people}\}$. We include one unary predicate $Drinks(p)$ where $p \in Drinks(p)$ iff the person $p$ drinks. Thus our structure will look like $A = (G, Drinks)$ and we can formalize the statement in FOL as follows

$$(\exists p)(Drinks(p) \rightarrow (\forall m)Drinks(m))$$

To show this is valid we need to show that this formula is always true for all structures with the appropriate vocabulary. We can do this by case analysis on the predicate $Drinks$. Since this is the only element of our vocabulary we can exhaustively look at all structures by examining this predicate. Every structure with this vocabulary must fall into one of three cases:

• $Drinks = \emptyset$
  In this case nobody in the group drinks. If this is the case then the formula is true since the first part of the implication will always be false.

• $Drinks = G$
  In this case everybody in the group drinks. If this is the case the formula will also be true since the latter part of the implication is always true.

• $Drinks \subset G$
  In this case the people that drink is a proper subset of the entire group. If this is the case then there is at least one person in the group that does not drink (since the relation is a proper subset). The formula will be valid because there will always exist a person for which the first part of the implication is false, and thus the formula will be true.