The phase transition in the random 1-3-HornSAT problem

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Abstract

This paper presents a study of the satisfiability of random Horn formulae and a search for a phase transition. This is a problem similar to the Satisfiability problem, but, unlike the latter, Horn satisfiability is tractable and thus it is easier to collect experimental data for large instances. We are also interested in Horn formulae because of their relation to finite automata. We study random Horn formulae generated according to a variation of the fixed-clause-length distribution model. Our experimental findings suggest that there is a sharp phase transition between a region where a random formula φ is almost surely satisfiable to a region where φ is almost surely unsatisfiable. We also use a result on random hypergraphs to generate a model that fits well our experimental data. This model though, suggests that the problem does not have a phase transition, showing how difficult it can be to establish experimentally a phase transition even for tractable problems like 1-3-HornSAT.

1 Introduction

In the past decade phase transitions in combinatorial problems have been studied intensively. Although the idea of phase transitions in combinato-

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rial problems was introduced as early as 1960 [17], in recent years it has been a main subject of research in the communities of theoretical computer science, artificial intelligence and statistical physics. Combinatorial phase transitions are also known as *threshold phenomena*. Phase transitions have been observed both on the probability that an instance of a problem has a solution (k-SAT, 3-Colorability) and on the computational cost for solving an instance (3-SAT, 3-Colorability). In few cases (2-SAT, 3-XORSAT, 1-in-k SAT) these phase transitions have been also formally proved [5, 12, 19, 15, 2, 8].

A problem that has been in the center of this research is that of 3-satisfiability (3-SAT). An instance of 3-SAT consists of a conjunction of clauses, where each clause is a disjunction of three literals. The goal is to find a truth assignment that satisfies all clauses. The *density* of a 3-SAT instance is the ratio of the number of clauses to the number of Boolean variables. We call the number of variables the *order* of the instance. Experimental studies [9, 32, 31] show that there is a shift in the probability of satisfiability of random 3-SAT instances, from 1 to 0, located at around density 4.26 (this is also called the *crossover point*). So far there is no proof of a sharp phase transition at that density, cf. [18, 14, 1]. The same experimental studies show a peak of the computational complexity around the crossover point. In [26], finite-size scaling techniques were used to demonstrate a phase transition at the crossover point. Later, in [6], further experiments showed that a phase transition of the running time from polynomial in the order to exponential is solver-dependent, and for several different solvers this transition occurs at density lower than the crossover point. A limitation on all the experimental studies is imposed by the inherent difficulty of the problem, especially around the crossover point. We can only study instances of limited order (usually up to few hundreds) before the problems get too hard to be solved in reasonable time using the available computational resources.

A problem that is similar to random 3-SAT is that of the satisfiability of random Horn formulae, also called random Horn-SAT. A Horn formula in conjunctive normal form (CNF) is a conjunction of Horn clauses; each Horn clause is a disjunction of literals[1] of which at most one can be positive. Unlike 3-SAT, Horn-SAT is a tractable problem. The complexity of the Horn-SAT is linear in the size of the formula [13]. The linear complexity of Horn-SAT allows us to study experimentally the satisfiability of the problem for much bigger input sizes than those used in similar research on other problems like 3-SAT or 3-Colorability [21, 9, 32, 7].

[1] A positive literal is a variable; a negative literal is a negated variable.

An additional motivation for studying random Horn-SAT comes from the fact that Horn formulae are connected to several other areas of Computer Science and Mathematics [28]. In particular Horn formulae are connected to automata theory, as the transition relation, the starting state, and the set of final states of an automaton can be described using Horn clauses. For example, if we consider automata on binary trees (see definition below), then Horn clauses of length three can be used to describe its transition relation, while Horn clauses of length one can describe the starting state and the set of the final states of the automaton (we elaborate on that later). Then, the question about the emptiness of the language of the automaton can be translated to a question about the satisfiability of the formula. There is also a close relation between knowledge-based systems and Horn formulae, but we do not consider this relation in this work. Finally, there is a correspondence between Horn formulae and hypergraphs that we use to show how results on random hypergraphs relate to our research on random Horn formulae.

The probability of satisfiability of random Horn formulae generated according to a variable-clause-length model has been studied by Istrate in [23]. In this work it is shown that according to this model random Horn formulae have a coarse satisfiability threshold, i.e. the problem does not have a phase transition. The variable-clause-length distribution model used by Istrate is better suited if we study Horn formulae in connection to knowledge-based systems [28].

Motivated by the connection between the automata emptiness problem and Horn satisfiability, we studied the satisfiability of two types of random Horn formulae in conjunctive normal form (CNF) that are generated according to a variation of the fixed-clause-length distribution model. That is, formulae that consist of clauses of length one and three only, and formulae that consist of clauses of length one and two only. We call these problems 1-3-HornSAT and 1-2-HornSAT respectively. We are looking to identify regions in the problems' space where instances are almost surely satisfiable or almost surely unsatisfiable. We are also interested in finding if the problems exhibit a phase transition, i.e. a sharp threshold.

Notice that the random 1-2-HornSAT problem is related to the random 1-3-HornSAT problem in the same way that random 2-SAT is related to random 3-SAT. That is, as some algorithm searches for a satisfying truth assignement for a random 1-3-Horn formula by assigning truth values to the variables, a random 1-2-Horn formula is created as a subformula of the original formula. This is a result of 3-clauses being shortened to 2-clauses by a subtitution of truth values. The relation between random 2-SAT and random 3-SAT is exploited by Achlioptas in [1] to improve on the lower bound for the threshold of random 3-SAT. In this work, Achlioptas uses differential equations to analyze the execution of a broad family of SAT algorithms. In general, one can try to analyze phase transitions using differential equations, cf. [22] The 1-2-HornSAT problem can be analyzed with the help of random graphs [4]. We show how results on random digraph connectivity, presented by Karp in [24], can be used to model the satisfiability of random 1-2-Horn formulae. These results can be used to show that there is no phase transition for 1-2-HornSAT and are matched by our experimental data.

Our experimental investigation of 1-3-HornSAT shows that there are regions where a random 1-3-Horn formula is almost surely satisfiable and regions where is almost surely unsatisfiable. Analysis of the satisfiability percentiles' window and finite-size scaling [33] suggest that there is a "sharp threshold line" between these two regions. As 1-2-HornSAT can be analyzed using random digraphs, 1-3-HornSAT can be analyzed using random hypergraphs. We show that some recent results on random hypergraphs [11] fit well our experimental data. Unlike the data analysis, the hypergraph-based model suggests that the transition from the satisfiable to unsatisfiable regions is rather a steep function than a step function. It is therefore not clear if the problem exhibits a phase transition, even though we were able to get experimental data for instances of large order.

Our work here also relates to that of Kolaitis and Raffill in [27]. There, the authors carried out a search for a phase transition in another NP-complete problem, that of AC-matching. The similarity between their work and ours is that the experimental data provide evidence that both problems have a slowly emerging phase transition. The difference is that in our case, because of the linear complexity of Horn satisfiability, we are able to test instances of Horn satisfiability of much bigger size, than the instances of AC-matching in [27] or actually most of the NP-complete problems like 3-SAT, 3-colorability etc.

2 Preliminaries

Let us review some definitions[2] related to combinatorial phase transitions. Let X be a finite set and $|X| = n$. Let A be a random subset of X constructed by a random procedure according to the probability space $\Omega(n,m) \stackrel{D}{=} (2^X, 2^{2^X}, \Pr)$, where Pr is defined as :

[2] The definitions found in this paper as well as more definitions and results can be found in [10]

$$\Pr_{\Omega(n,m)}(A) = \begin{cases} 1/\binom{n}{m^*} & \text{if } |A| = m^* \\ 0 & \text{otherwise} \end{cases},$$

where m is an integer and

$$m^* = \begin{cases} 0 & \text{if } m < 0 \\ m & \text{if } 0 \leq m \leq n \\ n & \text{if } m > n \end{cases}$$

The random procedure consists of selecting m^* elements of X without replacement. A (set) property Q of X is a subset of 2^X. Q is increasing if $A \in Q$ and $A \subseteq B \subseteq X$ implies $B \in Q$. Q is non-trivial if $\emptyset \notin Q$ and $X \in Q$. A property sequence Q consists of a sequence of sets $\{X_n : n \geq 1\}$ such that $|X_n| < |X_{n+1}|$ and a family $\{Q_n : n \geq 1\}$ where each Q_n is a property of X_n. Q is increasing (non-trivial) if Q_n is increasing (resp. non-trivial) for every $n \geq 1$.

Let Q_n be an increasing non-trivial property sequence $\theta : N \to R^+$ be a strictly positive function. We say that θ is a threshold for Q if for every $f : N \to N$:

1. If $\lim_{n\to\infty} f(n)/\theta(n) = 0$ then $\lim_{n\to\infty} \Pr_{\Omega(n,f(n))}(Q_n) = 0$
2. If $\lim_{n\to\infty} f(n)/\theta(n) = \infty$ then $\lim_{n\to\infty} \Pr_{\Omega(n,f(n))}(Q_n) = 1$

θ is a sharp threshold Q if for every $f : N \to N^+$:

1. If $\sup_{n\to\infty} f(n)/\theta(n) < 1$ then $\lim_{n\to\infty} \Pr_{\Omega(n,f(n))}(Q_n) = 0$
2. If $\inf_{n\to\infty} f(n)/\theta(n) > 1$ then $\lim_{n\to\infty} \Pr_{\Omega(n,f(n))}(Q_n) = 1$

We say that Q exhibits a phase transition if it has a sharp threshold. Our interest is in satisfiability of Horn formulas. Thus, in our framework X_n is the set of Horn clauses over a set with n Boolean variables. A set of Horn clauses is a Horn formula.

Our main motivation for studying the satisfiability of Horn formulae is that, unlike 3-SAT, this problem is tractable. Therefore we will have data for instances of much larger order to help us answer questions similar to those previously asked about 3-SAT.

Apart from that, it is of interest to us that Horn formulae can be used to describe finite automata. A finite automaton A is a 5-tuple $A = (S, \Sigma, \delta, s, F)$, where S is a finite set of states, Σ is an alphabet, s is a starting state, $F \subseteq S$ is the set of final (accepting) states and δ is a transition relation.

In a word automaton, δ is a function from $S \times \Sigma$ to 2^S. In a binary-tree automaton δ is a function from $S \times \Sigma$ to $2^{S \times S}$. Intuitively, for word automata δ provides a set of successor states, while for binary-tree automata δ provides a set of successor state pairs. A run of an automaton on a word $a = a_1 a_2 \cdots a_n$ is a sequence of states $s_0 s_1 \cdots s_n$ such that $s_0 = s$ and $(s_{i-1}, a_i, s_i) \in \delta$. A run is succesful if $s_n \in F$; in this case we say that A accepts the word a. A run of an automaton on a binary tree t labeled with letters from Σ, is a binary tree r labeled with states from S such that $\mathrm{root}(r) = s$ and for a node i of t, $(r(i), t(i), r(\text{left-child-of-}i), r(\text{right-child-of-}i)) \in \delta$. Thus, each pair in $\delta(r(i), t(i))$ is a possible labeling of the children of i. A run is succesfull if for all leaves l of r, $r(l) \in F$; in this case we say that A accepts the tree t. The language $L(A)$ of a word (resp. tree) automaton A, is the set of all words a (resp. trees t) for which there is a successful run of A on a (resp. t). An important question on automata theory that also is of great practical importance in the field of formal verification [34] is, given an automaton A is $L(A)$ non-empty ? We can show how the problem of non-emptiness of automata languages translates to Horn satisfiability.

Consider first a word automaton $A = (S, \Sigma, \delta, s_0, F)$. Construct a Horn formula φ_A over the set S of variables as follows:

- create a clause $(\bar{s_0})$
- for each $s_i \in F$ create a clause (s_i)
- for each element (s_i, a, s_j) of δ create a clause $(\bar{s_j}, s_i)$,

where $(s_i, \cdots, s_k)$ represents the clause $s_i \vee \cdots \vee s_k$ and $\bar{s_j}$ is the negation of s_j.

Theorem 1 *Let A be a word automaton and φ_A the Horn formula constructed as described above. Then $L(A)$ is non-empty if and only if φ_A is unsatisfiable.*

Proof.
($\Rightarrow$) Assume that $L(A)$ is non-empty, i.e. there is a path $\pi = s_{i_0} s_{i_1} \cdots s_{i_m}$ in A such that $s_{i_0} = s_0$ and $s_{i_m} = s_k$ where s_k is a final state. Since s_k is a final state (s_k) is a clause in φ_A. Also $(\bar{s_k}, s_{i_{m-1}})$ is a clause in φ_A. For φ_A to be satisfiable s_k should be true and consequently, $s_{i_{m-1}}$ must be true. By induction on the length of the path π we can show that for φ_A to be satisfiable s_0 must be true, which is a contradiction.
($\Leftarrow$) Assume that φ_A is unsatisfiable. It then must have positive-unit resolution refutation [20], i.e. a proof by contradiction where in each step one

of the resolvents must be a positive literal, where the last resolution step is with the clause $(\bar{s_0})$. Let (s_i) be the first positive literal resolvent in the proof. By construction, s_i is a final state of A. By induction on the length of the refutation, we can construct a path in A from s_0 to s_i, Therefore, $L(A)$ is non-empty. $\square$

Similarly to the word automata case, we can show how to construct a Horn formula from a binary-tree automaton. Let $A = (S, \Sigma, \delta, s_0, F)$ be a binary-tree automaton. Then we can construct a Horn formula φ_A using the construction above with the only difference that since δ in this case is a function from $S \times \{\alpha\}$ to $S \times S$, for each element (s_i, α, s_j, s_k) of δ, we create a clause $(\bar{s_j}, \bar{s_k}, s_i)$. It is not difficult to see that also in this case we have:

Theorem 2 *Let A be a binary-tree automaton and φ_A the Horn formula constructed as described above. Then $L(A)$ is non-empty if and only if φ_A is unsatisfiable.*

Motivated by the connection between tree automata and Horn formulas described in Theorem 2 we studied the satisfiability of two types of random Horn formulae. More precisely, let $H^{1,2}_{n,d_1,d_2}$ denote a random formula in CNF over a set of variables $X = \{x_1, \cdots, x_n\}$ that contains:

- a single negative literal chosen uniformly among the n possible negative literals
- $d_1 n$ positive literals that are chosen uniformly, independently and without replacement among all $n-1$ possible positive literals (the negation of the single negative literal already chosen is not allowed)
- $d_2 n$ clauses of length two that contain one positive and one negative literal chosen uniformly, independently and without replacement, among all $n(n-1)$ possible clauses of that type.

We call the number of variables n the *order* of the instance.

Let also $H^{1,3}_{n,d_1,d_3}$ denote a random formula in CNF over the set of variables $X = \{x_1, \cdots, x_n\}$ that contains:

- a single negative literal chosen uniformly among the n possible negative literals
- $d_1 n$ positive literals that are chosen uniformly, independently and without replacement among all $n-1$ possible positive literals (the negation of the single negative literal already chosen is not allowed)

- d_3n clauses of length three that contain one positive and two negative literals chosen uniformly, independently and with replacementamong all $\frac{n(n-1)(n-2)}{2}$ possible clauses of that type.

The sampling spaces $H^{1,3}$ and $H^{1,2}$ are slightly different; we sample with replacement in the first, and without replacement in the second. We explain here why. Assume that we sample dn clauses out of N uniformly at random with replacement. Let us consider the (asymptotic) expected number of distinct clauses we get. Each one of the N clauses will be chosen with probability $1-(1-\frac{1}{N})^{dn}$. The expected number of distinct chosen clauses is $N(1-(1-\frac{1}{N})^{dn})$. Notice that $N(1-(1-\frac{1}{N})^{dn}) \approx N(1-\exp\frac{-dn}{N}) = N(1-(1-\frac{dn}{N}+O((\frac{dn}{N})^2))) \approx dn - O(\frac{(dn)^2}{N})$. In the case of a random $H^{1,3}_{n,d_1,d_3}$ formula $N = \frac{n(n-1)(n-2)}{2}$ and clearly the expected number of distinct clauses we sample is asymptotically equivalent to dn; thus we sample with replacement for experimental ease. In the case of a random $H^{1,2}_{n,d_1,d_2}$ formula, we sample without replacement to ensure that we do not have many repetitions among the chosen clauses.

3 1-2-HornSAT

In this section we present our results on the probability of satisfiability of random 1-2-Horn formulae. We first present an experimental investigation of the satisfiability on the $d_1 \times d_2$ quadrant. We then discuss the relation between random 1-2-Horn formulae and random digraphs and show that our data agree with analytical results on graph reachability presented in [24].

We studied the probability of satisfiability of $H^{1,2}_{n,d_1,d_2}$ random formulae in the $d_1 \times d_2$ quadrant. We generated and solved 1200 random instances of order 20000 per data point. See Figure 1 where we plot the average probability of satisfiability against the two input parameters d_1 and d_2 (left) and the corresponding contour plot (right).

The satisfiability plot shown in Figure 1 indicates that the problem does not have a phase transition. This can also been observed if we fix the value of one of the input parameters. See Figure 2, where we show the satisfiability plot for random 1-2-HornSAT for various order values ranging from 500 to 32000, and for fixed $d_1 = 0.1$. We now explain why random 1-2-HornSAT does not have a phase transition, based on known results on random digraphs.

There are two most frequently used models of random digraphs. The first one, $G(n,m)$ consists of all digraphs on n vertices having m edges; all

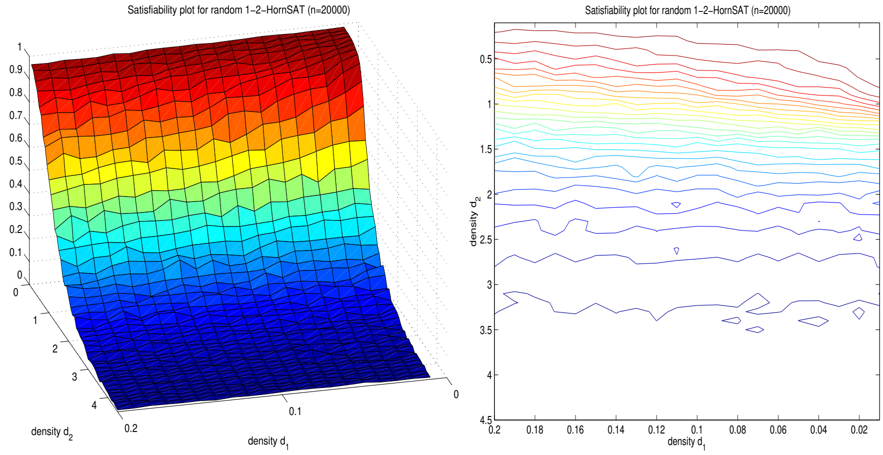

Figure 1: Average satisfiability plot of a random 1-2-Horn formula of order=20000 (left) and the corresponding contour plot (right).

digraphs have equal probability. The second model, $G(n, p(\text{edge}) = p)$ with $0 < p < 1$, consists of all digraphs on n vertices in which the edges are chosen independently with probability p. It is known that in most investigations the two models are interchangeable, provided certain conditions are met. In what follows, we will take advantage of this equivalence in order to show how our experimental results relate to analytical results on random digraphs [24].

We will first show that there is a relation between the satisfiability of a random $H^{1,2}_{n,d_1,d_2}$ formula and the vertex reachability of a random digraph $G(n, d_2 n)$. Let $\varphi \in H^{1,2}_{n,d_1,d_2}$, $(\bar{x_0})$ be the unique single negative literal in φ, and F be the set of all variables that appear as single positive literals in φ. Obviously $|F| = d_1 n$. Construct a graph G_φ such that for every variable x_i in φ there is a corresponding node v_i in G_φ and for each clause $(\bar{x_i}, x_j)$ of φ there is a directed edge in G_φ from v_i to v_j. G_φ is a random digraph from the $G(n, d_2 n)$ model.

It is not difficult to see that φ is unsatisfiable if and only if the node v_0 in G_φ is reachable from a node v_i such that $x_i \in F$. In other words, the probability of unsatisfiability of a random $H^{1,2}_{n,d_1,d_2}$ formula φ, is equal to the probability that a vertex of the random digraph $G(n, d_2 n)$ is reachable from a set[3] of vertices of size $d_1 n$.

[3] A vertex is reachable from a set of vertices if it is reachable by at least one of the vertices of the set.

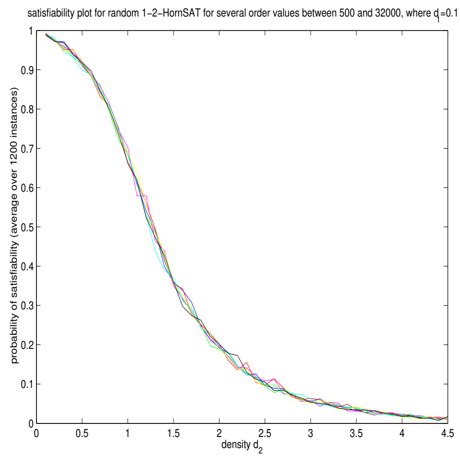

Figure 2: Satisfiability plot of random 1-2-Horn formulae when $d_1 = 0.1$

As mentioned above the $G(n, m)$ and $G(n, p((edge) = p))$ models can be used interchangeably, when $m \approx \binom{n}{2}p$ [4]. Therefore, the relation we established between the satisfiability of a random $H^{1,2}_{n,d_1,d_2}$ formula φ and the vertex reachability of a random digraph $G(n, d_2 n)$, holds also between φ and a random digraph $G(n, p = \frac{d_2}{n})$.

The vertex reachability of random digraphs generated according to the model $G(N, p)$ has been studied and analyzed by Karp in [24]. We use his results to study the satisfiability of random $H^{1,2}_{n,d_1,d_2}$ formulae. Karp showed that as n tends to infinity, when $np < 1 - h$, where h is a fixed small positive constant, the expected size of a connected component of the graph is bounded above by a constant $C(h)$. When $np > 1 + h$, as n tends to infinity, the set of vertices reachable from one vertex is either "small" (expected size bounded above by $C(h)$) or "large" (size close to Θn, where Θ is the unique root of the equation $1 - x - e^{-(1+h)x} = 0$ in $[0, 1]$). Moreover, a "giant" strongly connected component emerges of size approximately $\Theta^2 n$.

Let us now consider the two cases; $d_2 = 1 - h$ and $d_2 = 1 + h$, where h is a positive number. Remember that in our case $p = \frac{d_2}{n}$. In the analysis below we use the notation w.h.p. (with high probability) as shorthand for "with probability tending to 1 at the limit".

In the case where $d_2 = 1 - h$, that is $np < 1 - h$, the size of the set $X(v_i)$ of vertices reachable by a vertex v_i is w.h.p. less than or equal to $3 \ln n h^{-2}$, and the expected size of this set is bounded above by a constant related to h. Thus we get that the probability that v_0 is reachable by v_i

w.h.p. lies in the interval $[0, \frac{3 \ln n}{n(1-d_2)^2}]$, and its expected value is bounded above by a constant. The expected probability that v_0 is reachable by a set of $d_1 n$ vertices should increase with d_1. See the plots in Figures 1 and 2, which show that the probability of satisfiability of φ (which is 1 minus the probability that v_0 is reachable by a set of $d_1 n$ vertices in G_φ), while $d_2 < 1$, is decreasing as we increase d_2 and/or d_1.

When $d_2 = 1 + h$, that is $np > 1 + h$, we know that the set $X(v_i)$ of vertices reachable by a vertex v_i is w.h.p. either in the interval $[0, \frac{3 \ln n}{(1-d_2)^2}]$, or around Θn. We also know that the probability that $X(v_i)$ is "small" tends to $1 - \Theta$. Therefore, w.h.p. at least one of the $d_1 n$ vertices will have a "large" reachable set. That is, the probability that v_0 is reachable by a set of $d_1 n$ vertices is bounded below from Θ. Notice that Θ increases with d_2. Again, see the plots in Figures 1 and 2, where we can see that the probability of satisfiability of φ when $d_2 > 1$ is decreasing as d_2 increases. So the experimental observations are in agreement with the expectations based on the digraph reachability analysis.

Going back to digraphs' reachability, Karp's results show that for each vertex the set of its reachable vertices is very small up to the point where $np = 1$. We can observe the same behaviour in 1-2-HornSAT if we change our distribution model by setting $d_1 = c/n$ for some constant c. By doing that, we are adjusting our model to fit the reachability analysis done by Karp that is based on a single starting vertex in the digraph. The result of this modification is that d_1 is no longer a factor on the probability of satisfiability of φ, that depends now solely on d_2. See Figure 3, where we show the satisfiability plot in that case, and contrast with the picture that emerges when d_1 is a constant (shown in Figure 2). While before the satisfiability probability was steadily decreasing as we increased d_2, now the satisfiability probability is practically 1, until d_2 gets a value bigger than one. In both cases, however, the reachability analysis and the experimental data show that the satisfiability of random 1-2-Horn formulae is a problem that lacks a phase transition.

Remark 1 *One of the referees pointed out that the probability of satisfiability of 1-2-Horn can be calculated exactly. Using the combinatorics of labelled trees, one can calculate exactly the probability $P(k)$ that a given vertex v has an out-tree of size k, not including itself, in a random digraph with mean out-degree d_2. This is*

$$P(k) = \frac{e^{-(k+1)d_2} d_2^k}{k!} (k+1)^{k-1}$$

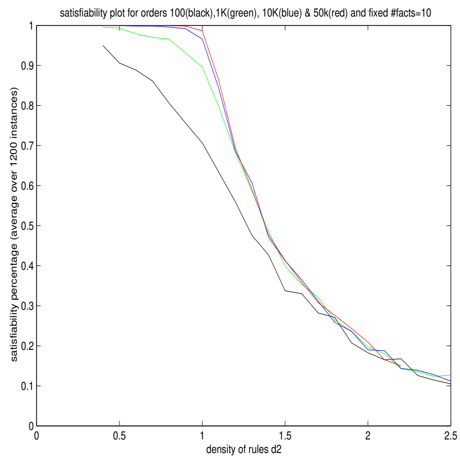

Figure 3: Satisfiability plot of random 1-2-Horn formulae when $d_1 = 10/n$ for orders 100(lower curve), 1K, 10K and 50K(higher curve).

The probability of satisfiability is then

$$P[\mathrm{SAT}] = \sum_{k=0}^{\infty} P(k)(1-d_1)^k$$

Numerical computation indicates a close fit with our experimental results.

4 1-3-HornSAT

In this section we present our results on the probability of satisfiability of random 1-3-Horn formulae. We first present a thorough experimental investigation of the satisfiability on the $d_1 \times d_3$ quadrant. We then show that analytic results on vertex identifiability in random hypergraphs [11] fit well our results on the satisfiablity of random 1-3-Horn formulae.

We studied the probability of satisfiability of $H^{1,3}_{n,d_1,d_3}$ random formulae in the $d_1 \times d_3$ quadrant. We generated and solved 3600 random instances of order 20000 per data point. See Figure 4 where we plot the average probability of satisfiability against the two input parameters d_1 and d_3 (left) and the corresponding contour[4] plot (right).

[4]In this plot there are 25 lines that separate consecutive percentages intervals, i.e. $[0\% - 4\%), [4\% - 8\%), \cdots, [96\% - 100\%]$.

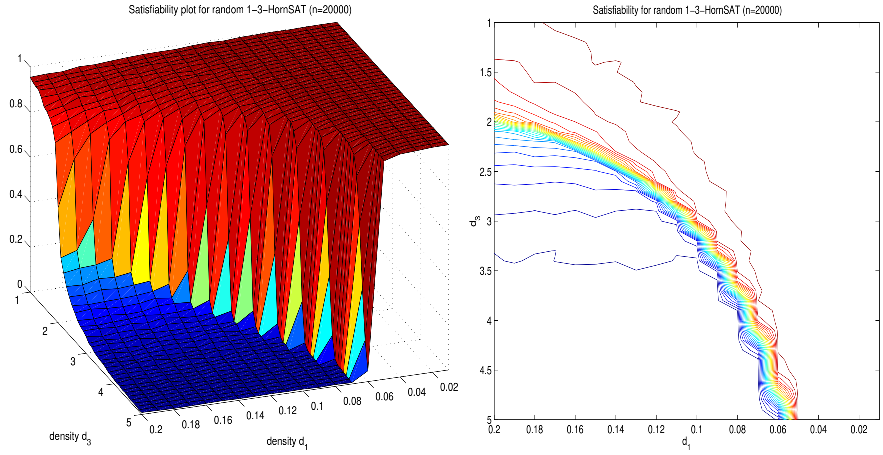

Figure 4: Average satisfiability plot of a random 1-3-Horn formula of order=20000 (left) and the corresponding contour plot (right).

From our experiments we see that there is a region where the formula is underconstrained (small values of d_1 and d_3) and the probability of satisfiability is almost 1. As the values of the two input parameters increase, there is a rapid change in the satisfiability terrain, what we call the *waterfall*. As the values of d_1 and d_3 cross some boundaries (the projection of the waterfall shown in the contour plots) the probability of satisfiability becomes almost 0. In other words, we observe a transition similar to these observed in other combinatorial problems like 3-SAT, 3-coloring etc.

There is a significant difference though, between these previously studied transitions and the one we observe in 1-3-HornSAT. In cases like 3-SAT or 3-colorability there are two input parameters describing a random instance; the order and the constrainedness (also called density in 3-SAT, and connectivity in 3-colorability) of the instance. The constrainedness is defined as the ratio of clauses for 3-SAT (or edges for 3-colorability) over variables (resp. vertices). In random 1-3-HornSAT, there are three parameters: the order of the instance and the two densities, namely d_1 and d_3. By taking a cut along the three dimensional surface shown in Figure 4 (left), we can study the problem as if it had only two input parameters.

We took two straight line cuts of the surface. For the first cut, we fixed d_1 to be 0.1, we let d_3 take values in the range $[1, 5.5]$ with step 0.1, and we chose order values 500, 1000, 2500, 5000, 10000, 20000 and 40000. See Figure 5(left), where we plot the probability of satisfiability along this cut. This plot reveals a quick change on the probability of satisfiability as the input parameter d_3 passes through a critical value (around 3). One technique that has been used to support experimental evidence of a phase transition is finite-size scaling. It is a technique coming from statistical mechanics that has been used in studying the phase transitions of several NP-complete problems, as k-SAT and AC-matching [26, 27]. This technique uses data from finite size instances to extrapolate to infinite size instances. The transformation is based on a rescaling according to a power law of the form $d' = \frac{d-d_c}{d_c}n^r$, where d is the density, d' is the rescaled parameter, d_c is the critical value, n is the order of the instance and r is a scaling exponent. As a result, a function $f(d,n)$ is transformed to a function $f(d')$. We applied finite-size scaling to our data to observe the sharpness of the transition. We followed the procedure presented by Kolaitis et al. in [27]. Our analysis yields the following finite-size scaling transformation:

$$d' = \frac{d_3 - 3.0385}{3.0385}n^{0.4859}$$

We then superimposed the curves shown in Figure 5(left) rescaled according to this transformation. The result is shown in Figure 5(right). The fit appears to be very good around zero, where curves collapse to a single universal curve, but as we move away from it is getting weaker. In the plot, the universal curve seems to be monotonic with limits $\lim_{d' \to -\infty} f(d') = 1$ and $\lim_{d' \to \infty} f(d') = 0$. This evidence suggests that there is a phase transition near $d_3 = 3$ for $d_1 = 0.1$.

We repeated the same experiment and analysis with the second cut, a straight line cut along the diagonal of the $d_1 \times d_3$ quadrant. In this case our formal parameter is an integer i. An instance with input paremeter value i, corresponds to an instance with densities $d_1 = \frac{i}{200}$ and $d_3 = \frac{i}{10} + 1$. In this case, by making the two input parameters d_1 and d_3 dependent, we effectively reduce the input parameters of the problem from three, (d_1, d_3, n), to two, (i, n). We let i take values in the range $[1, 40]$ with step 1, and we chose order values 500, 1000, 2500, 5000, 10000, 20000 and 40000. See Figure 6(left) where we plot the probability of satisfiability along this cut. This plot, as the one for the previous cut, reveals a quick change on the probability of satisfiability as the input parameter i passes through a critical value (around 19). We again used finite-size scaling on these data, looking for further support of a phase transition. For this cut, the analysis yields the following transformation:

$$i' = \frac{i - 19.1901}{19.1901}n^{0.2889}$$

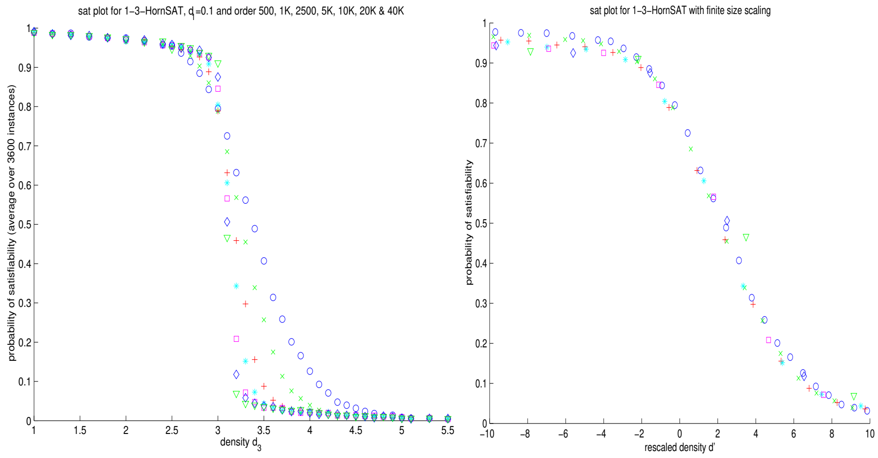

Figure 5: Average satisfiability plot of a random 1-3-Horn formula along the $d_1 = 0.1$ cut (left) and the satisfiability plot with rescaled parameter using finite-size scaling (right).

See Figure 6(right) where we superimpose the curves shown in the same figure (left) using the above transformation. As with the previous cut, the fit seems quite good, especially around zero, and the universal curve seems to have limits 1 and 0 in the infinities.

In our search for more evidence of a phase transition, we performed the following experiment for the cut used to produce the data in Figure 5 ($d_1 = 0.1$). For several values of order between 500 and 200000 and for density d_3 taking values in the range $[2.7, 3.8]$ with step 0.02, we generated and solved 1200 instances. We recorded for each different order value the values of density d_3 for which the average probability of satisfiability was 0.1, 0.2, 0.8 and 0.9 respectively[5]. The idea behind this experiment is that if the problem has a sharp threshold, i.e. a phase transition, then as the order of the instances increases the window between 10th and 90th probability percentiles, as well as that between the 20th and the 80th probability percentiles, should shrink and at the limit become zero. In Figure 7 we plot these windows. Indeed, they get smaller as the order increases.

Although Figure 7 shows that these windows indeed shrink as the order increases, it is not clear at all if at the limit they would go to zero. A further curve fitting analysis is more revealing. See Figure 8, where we plot the size of the 10%-90% probability of satisfiability window (left) and the

[5] We actually did linear regression on the two closest points to compute the density for each satisfiability percentage.

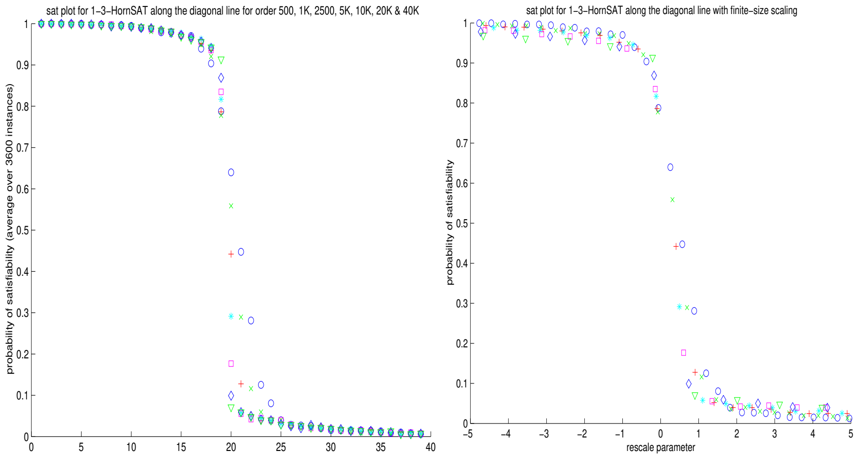

Figure 6: Average satisfiability plot of a random 1-3-Horn formula along the diagonal cut (left) and the satisfiability plot with rescaled parameter using finite-size scaling (right).

20%-80% probability of satisfiability window (right) as a function of the order. Using MATLAB to do curve fitting on our data, we find that both windows decrease almost as fast as $\frac{1}{\sqrt{n}}$. The correlation coefficient r^2 is almost 0.999, which gives a high confidence for the validity of the fit. This analysis suggests that indeed the two windows should be zero at the limit. That is an evidence that supports the existence of a phase transition for 1-3-HornSAT.

Similar analysis has been done before for k-SAT. The *width* of the satisfiability phase transition, which is the amount by which the number of clauses of a random instance needs to be increased so that the probability of satisfiability drops from $1-\epsilon$ to ϵ, is thought to grow as $\Theta(n^{1-\frac{1}{\nu}})$. Notice that the window that we estimate is equal to the normalized width (divided by the order). The exponent ν for $2 \leq k \leq 6$ is estimated in [25, 26, 29, 30]. It was also conjectured that as k gets large, ν tends to 1. Recently, Wilson in [35] proved that for all $k \geq 3$, $\nu \geq 2$, the transition width is at least $\Theta(n^{\frac{1}{2}})$. Our experiments suggest that the window of the satisfiability transition for 1-3-HornSAT shrinks as fast as $n^{-\frac{1}{2}}$, thus the transition width grows as $n^{\frac{1}{2}}$. We believe that the analysis in [35] can be applicable in the case of 1-3-HornSAT, and can complement our experimental findings.

In the rest of this section we will discuss the connection between random Horn formulae and random hypergraphs. We will show how recent results on random hypergraphs provide a good fit for our experimental data on

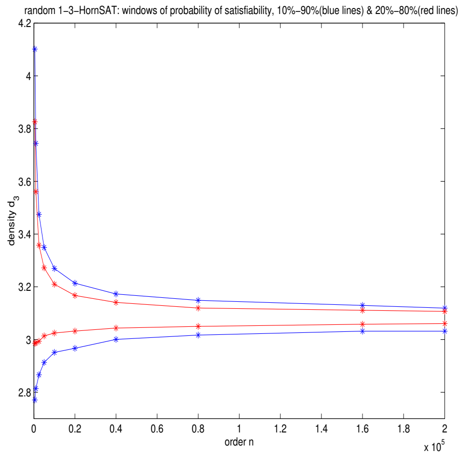

Figure 7: Windows of probability of satisfiability of random 1-3-Horn formulae along the $d_1 = 0.1$ cut. The outer two curves show the 10%-90% probability window, and the inner two curves show the 20%-80% probability window.

random 1-3-HornSAT presented so far. On the other hand, these results suggest that the transition is steep, but not a step function.

There is a one to one correspondence between random Horn formulae and random directed hypergraphs. Let φ be a $H^{1,3}_{n,d_1,d_3}$ random formula. We can represent φ with the following hypergraph G_φ [6]:

- represent each variable x_i in φ with a node v_i in G_φ
- represent each unit clause $\{x_k\}$ as a hyperedge in G_φ over v_k [7]
- represent each clause $\{x_j, \bar{x_k}, \bar{x_l}\}$ as a directed hyperedge in G_φ over the set $\{v_j, v_k, v_l\}$

In a recent development, Darling and Norris [11] proved some results on vertex identifiability in random undirected hypergraphs. A vertex v of a hypergraph is *identifiable in one step* if there is a hyperedge over v. A vertex v is *identifiable in* n steps if there is a hyperedge over a set S, such that $v \in S$ and all other elements of S are identifiable in less than n steps. Finally, a vertex v is *identifiable* if it is identifiable in n steps for some positive n.

[6] This representation actually ommitts the single negative literal that appears in φ.

[7] Hyperedges over vertices are called *patches* in [11] or *loops* in [16].

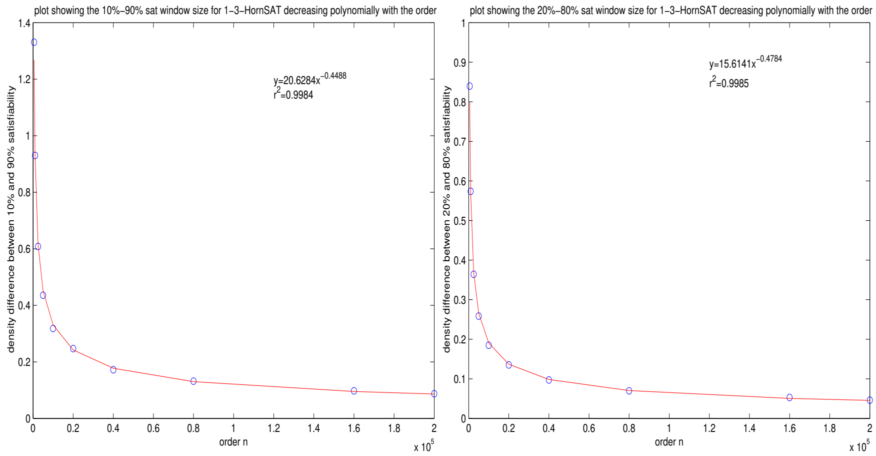

Figure 8: Plot of the 10%-90% probability of satisfiability window as a function of the order n (left) and of the 20%-80% probability of satisfiability window (right)

We now establish the equivalence between the satisfiability of φ and the *identifiability* of vertex v_k of G_φ, where $c = \{\overline{x}_k\}$ is the unique single negative literal clause of φ. First, we introduce an algorithm for solving Horn satisfiability.

We use a simple algorithm for deciding wether a Horn formula is satisfiable or not, presented by Dowling and Gallier in [13] (see also [3]). This algorithm runs in time $O(n^2)$ where n is the number of variables in the formula. Dowling and Gallier in their work actually describe how to improve this algorithm to run in linear time. For our purposes and for the sake of simplicity we use the simple quadratic algorithm.

Algorithm A.

begin

    let $\varphi = \{c_1, \cdots, c_m\}$
    consistent:=**true**; change:=**true**;
    set each variable x_i to be **false**;
    for each variable x_i such that $\{x_i\}$ is a clause in φ
        set x_i to **true**
    endfor;
    while (change **and** consistent) **do**
        change:=**false**;

```
        for each clause c_j in φ do
            if (c_j is of the form (x̄_1, ···, x̄_q)
              and all x_1, ···, x_q are set to true) then
                  consistent:=false;
            else
              if c_j is of the form {x_1, x̄_2, ···, x̄_q}
                and all x_2, ···, x_q are set to true
                and x_1 is set to false
                    then set x_1 to true; change:=true; φ := φ − c_j
              endif
            endif
        endfor
    endwhile
end
```

If Algorithm A terminates with consistent:=**true** then a satisfying truth assignement has been found. Otherwise, the formula φ is unsatisfiable.

Given a formula φ, its corresponding directed hypergraph G_φ, and a variable x_i, we prove the following relation between the truth value that Algorithm A assigns to x_i and the identifiability of vertex v_i of G_φ:

Lemma 1 *Algorithm A running on φ assigns the value* true *to x_i if and only if the vertex v_i of G_φ is identifiable.*

Proof. It is easy to show the equivalence by induction on the number of steps required to identify v_k (equivalently the number of iterations of the while loop of Algorithm A needed to set the value of x_k to **true**).
Base Case: If v_k is identifiable in one step, then $\{x_k\}$ is a clause in φ and Algorithm A immediately assigns the value **true** to it, and vice versa.
Inductive Hypothesis: A vertex is identifiable in $n-1$ steps if and only if the corresponding variable is set to **true** by Algorithm A in no more than $n-1$ iterations of the while loop.
Inductive Step: A vertex v_j that is identifiable in n steps, corresponds to a variable that appears in a clause of the form $\{v_j, \bar{v_{i_1}}, \cdots, \bar{v_{i_q}}\}$ and since all of $x_{i_1}, \cdots, x_{i_q}$ are already set to true, A will set x_j to **true** in the nth iteration of the while loop. Conversely, if x_j is set to **true** in the nth iteraton of the while loop of Algorithm A, then we derive that it appears in a clause of the form $\{x_j, \bar{x_{i_1}}, \cdots, \bar{x_{i_q}}\}$, where all of $x_{i_1}, \cdots, x_{i_q}$ are already set to **true**. But this implies that all $v_{i_1}, \cdots, v_{i_q}$ are identifiable in $n-1$ steps; therefore v_j is identifiable in n steps.
□

As an immediate result of this lemma we get:

Corollary 1 *Let φ be a $H^{1,3}_{n,d_1,d_3}$ random formula and $c = \{\bar{x_k}\}$ be the unique single negative literal clause of φ. Let G_φ be the directed hypergraph corresponding to φ. The formula φ is satisfiable if and only if the vertex v_k of G_φ is not identifiable.*

Darling and Norris in [11] studied the vertex identifiability in random undirected hypergraphs. Although Horn formulae correspond to directed hypergraphs, we decided to use the results of Darling and Norris in an effort to approximate the satisfiability of Horn formulae. The authors use the notion of a *Poisson random hypergraph.* A Poisson random hypergraph on a set V of n vertices with non-negative parameters $\{\beta_k\}_{k=0}^{\infty}$ is a random hypergraph Λ, where the numbers $\Lambda(A)$ of hyperedges of Λ over sets $A \subseteq V$ of vertices are independent random variables, depending only on $|A|$, such that $\Lambda(A)$ has distribution Poisson$(n\beta_k/\binom{n}{k})$, when $|A| = k$. Thus, the number of hyperedges of size k is Poisson$(n\beta_k)$, and they are distributed uniformly at random among all vertex sets of size k. (The Poisson distribution is a discrete distribution that takes on the values $X = 0, 1, 2, 3, \cdots$. The distribution is determined by a single parameter λ. The distribution function of the Poisson(λ) is $f(x) = \frac{\exp(-\lambda)\lambda^x}{x!}$. The expectation of Poisson(λ) is λ.) (Note that this model allows for more than one edge over a set $A \subseteq V$; for our purposes we only care if $\Lambda(A) = 0$ or not.)

One of the key results they proved is the following:

Theorem 3 [Darling-Norris] *Let $\beta = (\beta_j : j \in Z)$ be a sequence of non-negative parameters. Let $\beta(t) = \Sigma_{j\geq 0}\beta_j t^j$ and $\beta'(t)$ the derivative of $\beta(t)$. Let $z^* = \inf\{t \in [0,1) : \beta'(t) + \log(1-t) < 0\}$; if the infimum is not well-defined then let $z^* = 1$. Denote by ζ the number of zeros of $\beta'(t) + \log(1-t)$ in $[0, z^*)$.*

Assume that $z^* < 1$ and $\zeta = 0$. For $n \in N$, let V^n be a set of n vertices and let G^n be a Poisson(β) hypergraph on V^n. Then, as $n \to \infty$ the number V^{n*} of identifiable vertices satisfies the following limit w.h.p.: $V^{n*}/n \to z^*$.

If we ignore the direction[8] of the hyperedges then the random hypergraph G_φ representing a $H^{1,3}_{n,d_1,d_3}$ random formula corresponds to a Poisson(β) hypergraph G^n. To see that, notice that the hyperedges in G_φ are distributed

[8] Ignoring the direction of the hyperedges is equivalent to adding to the formula for each clause $(x \vee \bar{y} \vee \bar{z})$ two more clauses: $(\bar{x} \vee y \vee \bar{z})$ and $\bar{x} \vee \bar{y} \vee z$. Therefore we expect that the probability of satisfiability we get from the hypergraph model should be lower than the actual probability as it is measured by our experiments. This is indeed the case as we can see in Figure 10.

uniformly at random among all possible 1- and 3-sets of vertices, just like in a Poisson random hypergraph with only two non-zero parameters, β_1 and β_3. To find the values of these parameters, we set equal the probabilities that a hyperedge exists in the two hypergraphs G_φ and G^n. In G_φ, the probability that a variable x_i is selected as a positive unit literal is d_1. In G^n, the probability that there are zero hyperedges on x_i is e^{β_1}. From this we get $\beta_1 = -\log(1 - d_1)$. In G_φ, the probability that a 3-clause is selected (ignoring directions) is $nd_3/\binom{n}{3}$. In G^n, the probability that there are zero edges on the three variables in that clause is $e^{-n\beta_3/\binom{n}{3}} \approx 1 - n\beta_3/\binom{n}{3}$ (as $n \to \infty$). From this we get $\beta_3 = d_3$.

We used MATLAB (`www.mathworks.com`) to compute z^* for the hypergraph G^n on the quadrant $d_1 \times d_3$ [9]. From Corollary 1, we get that the probability of satisfiability of φ is 1 minus the probability that v_k is identifiable in G^n, which, by Theorem 3, is $1 - z^*$. See Figure 9(left) where we plot the probability of satisfiability of φ against the input parameters d_1 and d_3. A contour plot of the probability of satisfiability is given in Figure 9 (right).

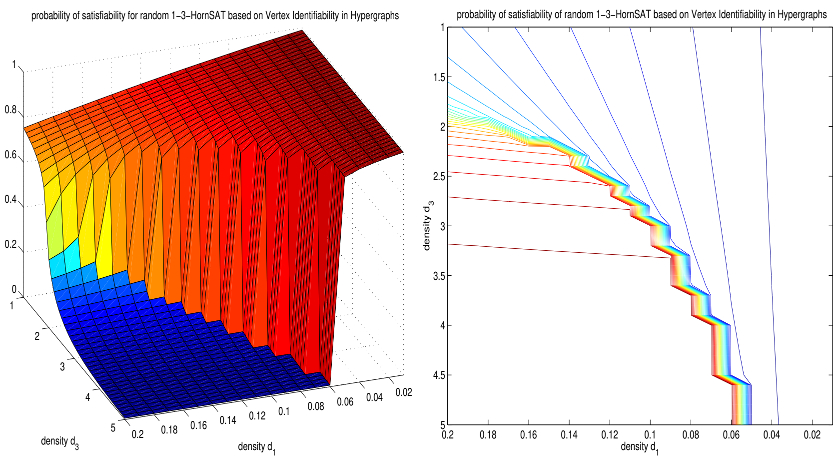

Figure 9: Probability of satisfiability plot of a random 1-3-Horn formula according to the vertex-identifiability model(left) and the corresponding contour plot (right).

Comparing the results derived by this model (Figure 9) and the results obtained by our experiments (Figure 4), we see that the model derived by the hypergraph analysis provides a very good fit of the experimental data. This is also obvious in Figure 10 where we plot the 50% satisfiability line according

[9] The Darling-Norris Theorem does not provide us an explicit result for z^*.

the model above (the rough curve) and according to our experimental data (smoother curve).

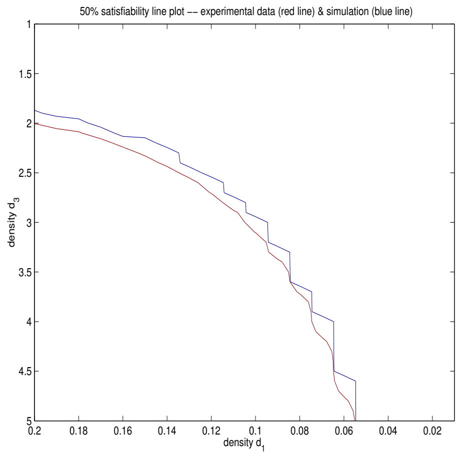

Figure 10: 50% satisfiability line – According to the model derived through hypergraphs (rough line) and according to our experimental data (smoother line).

Finally, we used our model to estimate the probability of satisfiability along the same two cuts that we presented earlier (the $d_1 = 0.1$ and the diagonal cut). See Figure 11 for the probability estimation along the two cuts according to the hypergraph-based model, and compare with our experimental findings shown in Figure 5 (left) and Figure 6 (left). For both cuts, the estimated probability has a steep drop that happens at the exact same point that the respective drop is observed in the experimental data. In Table 1 we give the raw data that correspond to the plots in Figure 11. Notice that, despite the very quick transition, the estimated curve is not a step function, as we would expect by looking our data and the limit curve after the finite-size scaling analysis (Figures 5 and 6 (right)). Should this be an accurate model for the 1-3-HornSAT, the probability of satisfiability is not be a step function at the limit, that is, the threshold function is not be a constant function.

$d_1 = 0.1$ cut		diagonal cut	
d_3	prob. of sat.	input parameter i	prob. of sat.
1	0.98775	1	0.99997
1.1	0.98619	2	0.99988
1.2	0.98455	3	0.9997
1.3	0.98282	4	0.99941
1.4	0.98098	5	0.99899
1.5	0.97903	6	0.99841
1.6	0.97694	7	0.99764
1.7	0.9747	8	0.99664
1.8	0.9723	9	0.99537
1.9	0.96969	10	0.99376
2	0.96685	11	0.99175
2.1	0.96372	12	0.98924
2.2	0.96026	13	0.98611
2.3	0.95637	14	0.98217
2.4	0.95194	15	0.97717
2.5	0.94679	16	0.97069
2.6	0.94062	17	0.96202
2.7	0.9329	18	0.94968
2.8	0.92244	19	0.9294
2.9	0.90522	20	0.072832
3	0.072832	21	0.063411
3.1	0.063588	22	0.055476
3.2	0.055745	23	0.048727
3.3	0.049039	24	0.042943
3.4	0.043267	25	0.038
3.5	0.038272	26	0.0335
3.6	0.033928	27	0.029856
3.7	0.030137	28	0.026559
3.8	0.026815	29	0.023665
3.9	0.023896	30	0.021117
4	0.021324	31	0.018868
4.1	0.019052	32	0.016878
4.2	0.017041	33	0.015114
4.3	0.015257	34	0.013547
4.4	0.013672	35	0.012153
4.5	0.012262	36	0.01091
4.6	0.011006	37	0.0098016
4.7	0.0098849	38	0.0088112
4.8	0.0088836	39	0.0079255
4.9	0.0079881	40	0.0071324

Table 1: Data for the probability of satisfiability of random 1-3-Horn formula according to the vertex-identifiability model, along the $d_1 = 0.1$ and the diagonal cut.

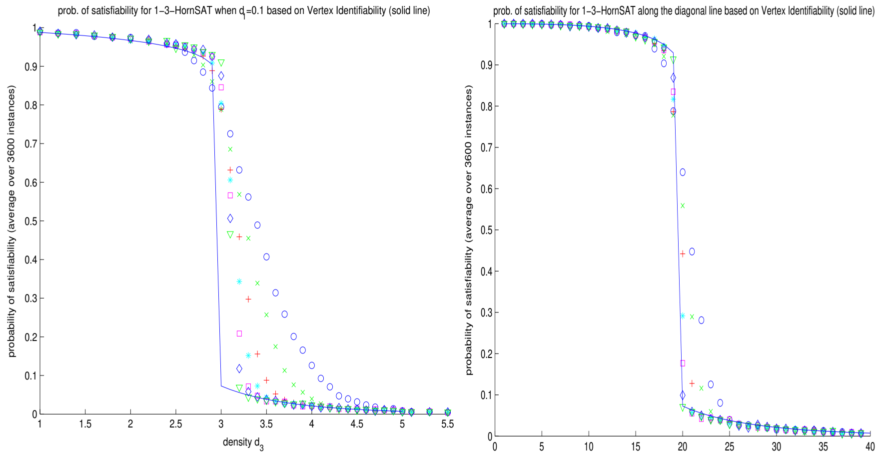

Figure 11: Probability of satisfiability plot of a random 1-3-Horn formula according to the vertex-identifiability model, along the $d_1 = 0.1$ cut (left) and the diagonal cut (right). The solid line corresponds to the model; the experimental data points are shown for comparison.

5 Conclusions

We set out to investigate the existence of a phase transition on the satisfiability of the random 1-3-HornSAT problem. This is a problem that is similar to 3-SAT, but its polynomial complexity allows us to collect data for much higher order.

We first showed, through our experimental findings and an analysis based on known results from digraphs' reachability, that the 1-2-HornSAT is a problem that lacks a phase transition.

On the contrary, our experiments provide evidence that the 1-3-HornSAT has a phase transition. By thoroughly sampling the $d_1 \times d_3$ quadrant, solving a large number of random instances of large order, we document a waterfall-like probability of satisfiability surface. In addition, by taking cuts of this surface, we are able to observe a quick transition from a satisfiable to an unsatisfiable region. When finite-size scaling is applied on these cuts, it suggests that there is a phase transition. Finally, analysis of the transition window provide further evidence for the phase transition.

We then used some recent results on random hypergraphs to generate a model for our experimental data. By comparing the waterfall-like probability surface against the estimated probability according to this model, we see that the hypergraph-based model fits well our experimental data. This suggests that further analysis based on hypergraphs could provide a rigorous analysis of the conjectured phase transition for the 1-3-HornSAT. This would be very significant since there are very few phase transitions that have been analytically proved (2-SAT, 3-XORSAT, 1-in-k SAT) [5, 12, 19, 15, 2, 8]. Although this model fits well our experimental data, when calculating the estimated probability along the two cuts, we see that the probability of satisfiability as the order goes to infinity is a very steep function, but not a step function. This last finding, which is contrary to our experimental findings, shows the difficulty of experimentally showing a phase transition, even for tractable problems such as 1-3-HornSAT.

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