The Decision Problem for the Probabilities of Higher-Order Properties

Phokion G. Kolaitis *and Moshe Y. Vardi
IBM Almaden Research Center

Abstract

The probability of a property on the class of all finite relational structures is the limit as $n \to \infty$ of the fraction of structures with n elements satisfying the property, provided the limit exists. It is known that *0-1 laws* hold for any property expressible in first-order logic or in fixpoint logic, i.e. the probability of any such property exists and is either 0 or 1. It is also known that the associated decision problem for the probabilities is PSPACE-complete and EXPTIME-complete for first-order logic and fixpoint logic respectively. The 0-1 law fails, however, in general for second-order properties and the decision problem becomes unsolvable.

We investigate here logics which on the one hand go beyond fixpoint in terms of expressive power and on the other possess the 0-1 law. We consider first *iterative logic* which is obtained from first order logic by adding **while** looping as a construct. We show that the 0-1 law holds for this logic and determine the complexity of the associated decision problem. After this we study a fragment of second order logic called *strict* Σ^1_1. This class of properties is obtained by restricting appropriately the first-order part of existential second-order sentences. Every strict Σ^1_1 property is NP-computable and there are strict Σ^1_1 properties that are NP-complete, such as 3-colorability. We show that the 0-1 law holds for strict Σ^1_1 properties and establish that the associated decision problem is NEXPTIME-complete. The proofs of the decidability and complexity results require certain combinatorial machinery, namely generalizations of Ramsey's Theorem.

1 Introduction

In recent years a considerable amount of research activity has been devoted to the study of the model theory of finite structures. This theory has interesting applications to several other areas including database theory [CH82,Va82] and complexity theory [Aj83, Gu84, Im83, Im86]. One particular direction of research has focused on the asymptotic probabilities of properties expressible in different languages and the associated decision problem for the values of the probabilities.

In general, if C is a class of finite structures over some vocabulary and if P is a property of some structures in C, then *the asymptotic probability* $\mu(P)$ on C is the limit as $n \to \infty$ of the

*On leave from Occidental College, Los Angeles, CA 90041.

fraction of the structures in C with n elements which satisfy P, provided that the limit exists. We say that P *is true almost everywhere* on C in case $\mu(P)$ is equal to 1. Combinatorialists have studied extensively the asymptotic probabilities of interesting properties on the class G of all finite graphs. It is, for example, well known and easy to prove that μ(connectivity)=1, while μ(l-colorabilty)=0, for any $l \geq 2$ [Bo79]. A theorem of Pósa [Po76] asserts that μ(Hamiltonicity)=1. The analysis of asymptotic probabilities can be useful when studying average-case complexity of algorithms.

Glebskii et al. [GKLT69] and Fagin [Fa76] were the first to establish a fascinating connection between logical definability and asymptotic probabilities. More specifically, they showed that if C is the class of all finite structures over some relational vocabulary and if P is any property expressible in first order logic, then $\mu(P)$ exists and is either 0 or 1. This result is known as *the 0-1 law for first order logic*. The proof of the 0-1 law also implies that the decision problem for the value of the probabilities of first order properties is solvable. Grandjean [Gr83] showed that the complexity of this problem is PSPACE-complete for bounded underlying vocabularies (i.e., for vocabularies with bounded arities). These results about almost sure truth are in sharp contrast with Trakhtenbrot's [Tr50] classical theorem to the effect that the set of first order sentences which are true on *all* finite relational structures is unsolvable, assuming that the vocabulary contains at least one binary relation symbol.

Using 0-1 laws, one can compute asymptotic probabilities of properties without going through the combinatorial analysis. Since first order logic has very limited expressive power on finite structures (cf. [AU79]), researchers investigated asymptotic probabilities in logical languages which go beyond first order. One such extension is *fixpoint logic*, which is obtained from first order logic by adding the *least fixed point* operator for formulas with positive second order variables [Mo74,CH82]. Blass, Gurevich and Kozen [BGK85] proved that a 0-1 law holds for fixpoint logic on the class of all finite relational structures. Moreover, they showed that the decision problem for the probabilities of fixpoint properties is EXPTIME-complete for a bounded vocabulary and 2EXPTIME[1] complete for an unbounded vocabulary (an unbounded vocabulary contains relation symbols of arbitrarily large arities). Fixpoint logic can express queries which are not first order definable such as connectivity and acyclicity. On the other hand, the class of fixpoint queries is in general properly contained in PTIME. The "parity" query in particular ("there is an even number of elements") is not expressible in fixpoint logic over the class of all finite structures. Thus, one would like to consider even higher-order languages, such as second order logic. Kauffman and Shelah [KS85] studied asymptotic probabilities of properties expressible in monadic second order logic and found that the 0-1 law fails in a strong way for this logic. In addition, the set of almost surely true monadic second order sentences is unsolvable.

Our aim in this paper is to study languages which on the one hand go beyond fixpoint in terms of expressive power and on the other possess the 0-1 law.

We consider first *iterative logic*, which was introduced by Chandra and Harel [CH82] (who called it RQ). Iterative logic is obtained from first order logic by adding **while** looping as an iteration construct. The properties expressible in iterative logic contain all fixpoint queries and are in turn contained in the ones computable in PSPACE. Although it is still an open problem to show that iterative logic properly extends fixpoint logic, there is, however,

[1]The prefix 2 stands for "doubly". Thus, 2EXPTIME denotes the class of languages accepted by Turing machines that run in doubly exponential time, and similarly for 2EXPSPACE and 2NEXPTIME.

strong evidence to this effect, since all the properties expressible by fixpoint logic are in PTIME while iterative logic can express PSPACE-complete properties [CH82]. (In fact, on *ordered* finite structures, fixpoint logic can express *all* PTIME properties [Im86,Va82], while iterative logic can express *all* PSPACE properties [Va82].)

We prove here that a 0-1 law holds for properties expressible in iterative logic on the class of all finite relational structures, extending thus the result in [BGK85]. We examine the decision problem for the values of the probabilities of properties expressible in iterative logic both for bounded and unbounded vocabularies. We show that in iterative logic, as in [BGK85] for fixpoint logic, there is an exponential gap between the two cases. More specifically, the problem is EXPSPACE-complete for bounded vocabularies and 2EXPSPACE-complete for unbounded vocabularies. It turns out that such a gap exists for all the logics we consider here, e.g., first order logic and transitive closure logic.

After this, we investigate the asymptotic probabilities of properties expressible in fragments of second order logic. The simplest and most natural fragments of second order logic are formed by considering second order formulas with only existential second order quantifiers or with only universal second order quantifiers. These are the well known classes of Σ^1_1 and Π^1_1 formulas respectively. Fagin [Fa74] proved that the Σ^1_1 definable properties are exactly the NP-computable properties. Thus, on the class of all finite relational structures Σ^1_1 and Π^1_1 definability is a proper extension of definability in fixpoint logic. The 0-1 law, however, fails for Σ^1_1 properties in general (and consequently for Π^1_1 properties as well), since, for example, "parity" is definable by such a formula. Moreover, it is not hard to show that the Σ^1_1 properties having probability 1 form an unsolvable set.

In view of these facts, we concentrate on fragments of Σ^1_1 formulas in which we restrict the pattern of the first order quantifiers which occur in the formulas. If Ψ is a class of first order sentences, then we denote by $\Sigma^1_1(\Psi)$ the class of Σ^1_1 sentences where the first order part is in Ψ. Two remarks are in order now. First, if Ψ is the class of all $\exists^*\forall^*\exists^*$ first order sentences (that is to say, first order sentences whose quantifier prefix consists of a string of existential quantifiers, followed by a string of universal quantifiers, followed by a string of existential quantifiers), then $\Sigma^1_1(\Psi)$ has the same expressive power as the full Σ^1_1. In other words, every Σ^1_1 formula is equivalent to one of the form

$$\exists\overline{S}\exists\overline{x}\forall\overline{y}\exists\overline{z}\theta(\overline{S},\overline{x},\overline{y},\overline{z}),$$

where θ is a quantifier free formula, $\overline{S}$ is a sequence of second order relation variables and $\overline{x},\overline{y},\overline{z}$ are sequences of first order variables (*Skolem normal form*). Second, it follows from Trakhtenbrot's [Tr50] theorem, that for any class Ψ the decision problem for $\Sigma^1_1(\Psi)$ properties having probability 1 is at least as hard as the finite satisfiability problem for sentences in Ψ. This problem is already unsolvable even in the case where Ψ is the class of $\exists^*\forall^*\exists^*$ sentences (cf. [Le79]). As a result, in order to pursue positive results one has to consider fragments $\Sigma^1_1(\Psi)$, where Ψ is a class for which the finite satisfiability problem is solvable.

The class of *Bernays-Schönfinkel* formulas is the collection of all first order formulas with prefixes of the form $\exists^*\forall^*$. In his classical paper *On a Problem in Formal Logic* Ramsey [Ra28] established that the finite satisfiability problem for this class is solvable. Lewis [Le80] showed that the problem is actually NEXPTIME-complete.

We say that a formula is *strict* Σ^1_1 if it is in the class $\Sigma^1_1(\Psi)$, where Ψ is a Bernays-Schönfinkel formula (the term "strict Σ^1_1" is from [Ba75]) . Thus, every strict Σ^1_1 formula is of the form

$$\exists \overline{S} \exists \overline{x} \forall \overline{y} \theta(\overline{S}, \overline{x}, \overline{y}),$$

where θ is quantifier free. The strict Σ^1_1 formulas not only can express graph-theoretic properties such as disconnectivity, but they can also capture natural NP-complete problems, such as l-colorability, for any $l \geq 2$, monochromatic triangle, and others.

We prove here that a 0-1 law holds for strict Σ^1_1 properties (and therefore also for the strict Π^1_1 properties). It is, perhaps, worth pointing out that the proof uses preservation and compactness arguments and is quite different from the argument used for iterative logic.

We finally focus on the decision problem for the values of the probabilities of strict Σ^1_1 properties. Our main theorem here is that the problem is NEXPTIME-complete for bounded vocabularies and 2NEXPTIME-complete for unbounded vocabularies. Unlike iterative logic, the solvability of this decision problem does not follow from the proof of the 0-1 law for strict Σ^1_1. In order to establish the main result we have to use instead elaborate combinatorial machinery, namely Ramsey type theorems for *set systems* from Nešetřil and Rödl [NR77,NR83]). These are used to build certain finite *canonical* models from which in turn we can determine if a strict Σ^1_1 sentence has probability 1.

2 Random Structures

Let σ be a vocabulary consisting of relation symbols *only* and let C be a class of finite structures over σ. We assume that the universe of every structure in C is an initial segment $\{1, 2, ..., n\}$ of the integers and that C is closed under isomorphisms. If P is a property of (some) structures in C, then the *labeled asymptotic probabilty* $\mu(P)$ on C is defined to be equal to the limit as $n \to \infty$ of the fraction of structures in C of cardinality n which satisfy P, provided this limit exists. If L is a logic, we say that the 0-1 *law holds for L on C* in case $\mu(P)$ exists and is equal to 0 or 1 for every property P expressible in the logic L. The 0-1 law for a logic L on a class C is usually established as follows (following Fagin's proof of the 0-1 law for first order logic [Fa76]): one shows that there is an infinite structure $\mathbf{A}$ over the vocabulary σ such that for any property P expressible in L we have:

$$\mathbf{A} \text{ satisfies } P \Longleftrightarrow \mu(P) = 1 \text{ on C }. \tag{1}$$

It turns out that for the logics we consider here, namely first order, fixpoint, iterative, and strict Σ^1_1, there is a single countable structure $\mathbf{A}$ that satisfies the above equivalence for all of them. Moreover, this structure $\mathbf{A}$ is unique up to isomorphism. We call $\mathbf{A}$ the *countable random structure over the vocabulary* σ. The random structure $\mathbf{A}$ is characterized by an infinite set of *extension axioms*, which, intuitively, assert that every *type* can be *extended* to any other possible type. More precisely, if $\overline{x} = (x_1, ..., x_n)$ is a sequence of variables, then a *type* $t(\overline{x})$ *in the variables* $\overline{x}$ *over* σ is a conjunction of atomic and negated atomic formulas in $\overline{x}$ such that for every atomic formula $R(\overline{x})$ exactly one of $R(\overline{x})$ and $\neg R(\overline{x})$ occurs in $t(\overline{x})$. We say that *a type* $s(\overline{x}, z)$ *extends the type* $t(\overline{x})$ if every conjunct of t is also a conjunct of s. With each pair of types s and t such that s extends t we associate a first order *extension axiom* τ which states that

$$\forall \overline{x}(t(\overline{x}) \to \exists z(\bigwedge_i z \neq x_i \wedge s(\overline{x}, z))).$$

Let T be the set of all extension axioms. The theory T was studied by Gaifman [Ga64], who showed, using a *back and forth* argument, that any two countable models of T are isomorphic (*T is an ω-categorical theory*). Fagin [Fa76] realized that the extension axioms are relevant to the study of probabilities on finite structures and proved that on the class C of all finite structures of vocabulary σ

$$\mu(\tau) = 1$$

for any extension axiom τ. The equivalence (1) between truth on $\mathbf{A}$ and almost sure truth on C (and consequently the 0-1 law for first order logic on C) follows from these results by a compactness argument.

The decision problem for the values of the probabilities of first order properties on the class C of all finite structures over the vocabulary σ is now reduced to the problem of deciding the truth value of first order properties on the random structure $\mathbf{A}$. As mentioned earlier, Grandjean [Gr83] proved that the latter problem is PSPACE-complete, where the underlying vocabulary σ is assumed to be bounded (i.e., there is a some bound on the arity of the relation symbols in σ). This results depends heavily on the following two facts:

(a) The random structure $\mathbf{A}$ is *homogeneous*, that is, if $\overline{a}$ and $\overline{b}$ are two n-tuples from $\mathbf{A}$ satisfying the same type $t(\overline{x})$, then there is an automorphism of $\mathbf{A}$ mapping $\overline{a}$ to $\overline{b}$.

(b) If σ is a finite vocabulary with m relation symbols and such that the biggest arity among these symbols is d, then for every $n \geq 1$ there are at most 2^{mn^d} types in n variables (note that this bound is exponential if d is fixed, and doubly exponential otherwise). Moreover, if $t(\overline{x})$ is a type in n variables, then there are at most $2^{m(n+1)^d - mn^d}$ types $s(\overline{x}, z)$ in $(n+1)$ variables extending t.

The random structure $\mathbf{A}$ is shown to be homogeneous using a back and forth argument that builds the automorhism step by step with the help of the extension axioms. The homogeneity of $\mathbf{A}$ implies that for any type $t(\overline{x})$ and any first order formula $\psi(\overline{x})$ we have that either

$$\mathbf{A} \models \forall \overline{x}(t(\overline{x}) \rightarrow \psi(\overline{x})) \tag{2}$$

or

$$\mathbf{A} \models \forall \overline{x}(t(\overline{x}) \rightarrow \neg\psi(\overline{x})). \tag{3}$$

Intuitively, this says that *only types count* in determining the truth of a formula on $\mathbf{A}$. More formally, (2) and (3) above give rise to an alternating Turing machine (ATM) algorithm which on input $(t(\overline{x}), \psi(\overline{x}))$ determines which of the two cases (2) and (3) above holds on $\mathbf{A}$. The crucial step in the algorithm is when the formula ψ starts with an existential quantifier. In this case we have that

$$\mathbf{A} \models \forall \overline{x}(t(\overline{x}) \rightarrow \exists z \theta(\overline{x}, z)))$$

if and only if either for some x_i we have

$$\mathbf{A} \models \forall \overline{x}(t(\overline{x}) \rightarrow \theta(\overline{x}, x_i)))$$

or for some extension $s(\overline{x}, z)$ of $t(\overline{x})$ we have

$$\mathbf{A} \models \forall \overline{x} \forall z(s(\overline{x}, z) \rightarrow \theta(\overline{x}, z))).$$

The estimates in (b) above on the number of types in n variables and on the number of types extending a given type are used to show that this algorithm works in ATM polynomial time (=PSPACE) for bounded vocabularies. The lower bound follows from the fact that for any structure with two distinct elements the decision problem for truth of first order properties is PSPACE-hard (Stockmeyer [St76]).

3 Iterative Logic

It is well known that on finite structures first order logic has a limited expressive power, while at the same time it lacks the nice properties that one usually associates with first order logic, such as compactness and axiomatizability [Gu84]. In seeking higher expressive power, one is motivated to consider higher-order logics. We study here *iterative logic*, which is obtained from first order logic by adding **while** looping [CH82].

Let σ be a vocabulary of relation symbols (we exclude 0-ary relation symbols). We obtain σ' be adding to σ infinitely many relation variables X_i, $i \geq 0$.

To define the language, we first define *programs*.

- If φ is a first-order formula, $\overline{x}$ is a sequence of the free variables of φ, and X is an $|\overline{x}|$-ary relation variable, then $X(\overline{x}) \leftarrow \varphi$ is a program.
- If S_1 and S_2 are programs, then $(S_1; S_2)$ is a program.
- If χ is a first order sentence and S is a program, then **while** χ **do** S is a program.

Although this language may seem strange to the logician, it is quite natural if we view logic as a query language. Indeed, this language (in somewhat different syntax) was introduced in [CH82] as the relational database query language RQ. To understand the semantics of the language, think of a formula as an expression denoting a relation. Thus, $X(\overline{x}) \leftarrow \varphi$ can be viewed as an *assigment* statement. Iterative logic, therefore, consists of the canonical programming constructs of *assignment*, *sequencing*, and *looping*. Formally, the semantics of a program S is a function $\langle S \rangle : Str(\sigma') \to Str(\sigma')$, where $Str(\sigma')$ is the class of structures over σ'. The semantics of assignment, sequencing, and looping is defined in the standard way (if the loop does not terminate, then the result of the loop is the assigment of the empty relation to the relation variables). Details will be supplied in the full paper.

Finally, if S is a program and X is a relation variable, then $(X)S$ is a formula. Given a structure $\mathbf{B}$ on σ, we evaluate $(X)S$ as follows. Let $\mathbf{B}'$ be a structure on σ', such that $\mathbf{B}$ is the reduct of $\mathbf{B}'$ to σ, and all the X_i's are assigned the empty relation. Let $\mathbf{B}''$ be the result of applying S to $\mathbf{B}'$. Then $(X)S$ denote the relation assigned to X in $\mathbf{B}''$. Note that if X is a 0-ary relation, then $(X)S$ is a sentence.

We now prove the following:

Theorem 3.1: *The 0-1 law holds for iterative logic on the class C of all finite structures over a relational vocabulary σ. Moreover, if $\mathbf{A}$ is the countable random structure and P is a propery expressed in iterative logic, then*

$$\mathbf{A} \models P \iff \mu(P) = 1$$

Proof: The argument generalizes the argument in [BGK85] for fixpoint logic.

The 0-1 law for iterative logic uses a well known model-theoretic characterization of ω-categorical theories due to Ryll-Nardzewski [Ry59] and Vaught [Va61]. According to this, a theory has a unique (up to isomorphism) countable model if and only if for every n there are only finitely many inequivalent first order formulas with n variables in the models of T.

We consider iterative logic over the random structure $\mathbf{A}$, and show that looping can be eliminated, i.e., every formula in iterative logic is equivalent over $\mathbf{A}$ to a first order formula. For simplicity, consider a program **while** χ **do** S, where S is **while**-free. Suppose that S uses the variables $X_1, ..., X_k$. We can find first order formulas $\theta_i \sup j$, for $1 \leq i \leq k$ and $j \geq 0$, over σ such that $\theta_i \sup j$ describes the assignment to X_i after the j-th iteration of S.

It follows that there are integers M, N, $M < N$, such that the random structure $\mathbf{A}$ satisfies the first order sentences that asserts $\theta_i \sup M \equiv \theta_i \sup N$, for $1 \leq i \leq k$. Thus, if the program has not terminated after N steps, it is guaranteed to loop. This means that there is a first order formula ψ such that (X_j) **while** χ **do** S is equivalent to ψ on $\mathbf{A}$. Furthermore, the equivalence (1) above (where L is first order) implies that the formulas $\theta_i \sup M \equiv \theta_i \sup N$, for $1 \leq i \leq k$, are almost everywhere true on C and, consequently, (X_j) **while** χ **do** S is equivalent to ψ almost everywhere. The 0-1 law for iterative logic is now obtained immediately from the 0-1 law for first order and, moreover, the equivalence (1) extends to iterative logic. ∎

Consider now the decision problem for the probabilities of iterative logic properties, i.e., the decision problem for the set $\{\varphi : \varphi \text{ is an iterative logic sentence and } \mu(\varphi) = 1\}$. The proof of the 0-1 law for iterative logic shows that the problem is solvable, since by enumerating proofs we can find M and N, $M < N$ such that $\theta_i \sup M \equiv \theta_i \sup N$, for $1 \leq i \leq k$. Thus, we can find ψ and check whether $\mu(\psi) = 1$. Unfortunately, this does not give us any complexity bounds.

To get upper bounds, we use the fact from Section 2 that *only types count* in determining the truth of first order formulas. Consider again the program **while** χ **do** S, where S is **while**-free and uses the variables $X_1, ..., X_k$. We know that we can find first order formulas $\theta_i \sup j$, for $1 \leq i \leq k$ and $j \geq 0$, over σ such that $\theta_i \sup j$ describes the assignment to X_i after the j-th iteration of S. Thus, we can emulate the computation of S on $\mathbf{A}$, by figuring out, using Grandjean's algorithm [Gr83], which types satisfy the formulas $\theta_i \sup j$.

Suppose that the maximal arity of the X_i's is e, and suppose that the S refers to m relation symbols from the underlying vocabulary with arity bounded by d. Then we know that there are at most $2 \sup me \sup d$ types in e variables. Thus, we need space $O(2 \sup me \sup d)$ to emulate S. Note that if d is fixed, then this bound is exponential, and if d is not fixed (and therefore is a parameter of the problem), then the above bound is doubly exponential.

Theorem 3.2: *The decision problem for the probabilities of iterative logic is EXPSPACE-complete for bounded vocabularies and 2EXPSPACE for unbounded vocabularies.*

Proof: We have sketched above the idea for the proof of the upper bounds. To prove the lower bounds we encode Turing machines that run in exponential (doubly exponential space). The idea is to describe an ID of the machine by a relation. That is, at every stage of the computation of **while** χ **do** S, the assigment to the relation variables encodes a description of the Turing machine. More precisely, if at a certain stage of the computation we have $X(\overline{a})$ for some tuple $\overline{a}$, then $\overline{a}$ encodes a cell number and a cell content. We encode cell numbers by types. With bounded vocabulary we can encode numbers up to 2^n, and with unbounded vocabulary we can encode numbers up to $2 \sup 2 \sup n$. ∎

Theorem 3.2 establishes an exponential gap between the bounded and the unbounded cases of the decision problem for iterative logic. The reason is that there are exponentially more types in the unbounded case. A similar gap was established in [BGK85] for fixpoint logic. We now show that such a gap exists for several other logics. The logics we consider in this section are *transitive closure* logic, first order logic, and fragments of first order logic.

Transitive closure logic is a fragment of fixpoint logic, where the least fixpoint operator can be used only to form the transitive closure of relation. For a precise definition see [Im83], where it is shown that over *ordered* finite structures, transitive closure logic can express all NLOGSPACE properties. In addition to first order logic, we also consider the fragments Σ_k^0 and Π_k^0, where we restrict the pattern of quantifiers to consist of at most k alternations.

We need the following notation for complexity classes, which refers to alternation as a computational resource [Be80]. Let $TA(f(n), g(n))$ be the class of languages accepted by alternating Turing machines in time $O(f(n))$ starting at an existential state and using at most $O(g(n))$ alternations. For example, $TA(poly, k)$ is Σ_k^{poly}, and $TA(poly, poly)$ is PSPACE [CKS81]. Let $TA(poly, lin)$ be the class of languages accepted by an alternating polynomial time machine with a linear number of alternations. Similarly, let $TA(exp, lin)$ be the class of languages accepted by alternating exponential time machines that use only a linear number of alternations.

Theorem 3.3: *The following table describes the complexity of the decision problems for the probabilities:*

Logic	*Bounded Vocabulary*	*Unbounded Vocabulary*
Σ_k^0	$TA(poly, k)$	$TA(exp, k)$
First Order Logic	$TA(poly, lin)$	$TA(exp, lin)$
Transitive Closure Logic	*PSPACE*	*EXPSPACE*
Fixpoint Logic	*EXPTIME*	*2EXPTIME*
Iterative logic	*EXPSPACE*	*2EXPSPACE*

The result for first order logic in the bounded case is from [Gr83], and in the unbounded case was announced by Immerman [BGK85]. The results for fixpoint logic are from [BGK85]. We note that the results for bounded vocabularies hold even for finite vocabularies, provided the arities of the relation variables are unbounded, or for bounded relation variables, provided the vocabulary is infinite.

Iterative logic can be viewed as an infinitary first-order logic with a finitary representation. In what follows we consider fragments of second order logic.

4 The 0-1 Law for Strict Σ_1^1 properties

In this section we prove that if P is any strict Σ_1^1 property, then the asymptotic probability of P exists and is either 0 or 1. This 0-1 law is an immediate consequence of Lemmas 4.1 and 4.3 below. Before stating and proving these lemmas we introduce some notation.

If $\mathbf{B}$ is a structure over the vocabulary σ and ψ is a sentence in some logic, then $\mathbf{B} \models \psi$ means that the sentence ψ is true on $\mathbf{B}$. We write $\models \psi$ to denote that ψ is true on *all* structures over the vocabulary σ. We also write $\models_{FIN} \psi$ in case ψ is true on *all finite* structures over σ. Recall that a Π^1_1 formula is one of the form $\forall \overline{S}\theta(\overline{S})$, where $\overline{S}$ is a sequence of second order relational variables and θ is an arbitrary first order formula. A *strict* Π^1_1 formula is the dual of a strict Σ^1_1 formula and so it is of the form $\forall \overline{S}\forall \overline{x}\exists \overline{y}\theta(\overline{S}, \overline{x}, \overline{y})$, where θ is a quantifier free formula.

Lemma 4.1: *Let* $\mathbf{A}$ *be the countable random structure over* σ *and let* $\forall \overline{S}\theta(\overline{S})$ *be an arbitrary* Π^1_1 *sentence. If* $\mathbf{A} \models \forall \overline{S}\theta(\overline{S})$*, then there is a first order sentence* ψ *over the vocabulary* σ *such that:*

$$\mu(\psi) = 1$$

and

$$\models \psi \to \forall \overline{S}\theta(\overline{S}).$$

In particular, every Π^1_1 *sentence that is true on* $\mathbf{A}$ *has probability 1 on* C*.*

Proof: Let T be the set of all the *extension axioms* for the random structure $\mathbf{A}$ and let

$$T^* = T \cup \{\neg\theta(\overline{S})\}.$$

Towards a contradiction, assume that the hypothesis of the lemma holds, but the conclusion fails. Since the conjunction of any finite number of the extension axioms is a first order sentence of probability 1, it follows that every finite subset of T^* has a model over the expanded vocabulary $\sigma^* = \sigma \cup \{\overline{S}\}$. By compactness, the set T^* has a countable model $\mathbf{B}$ over σ^*. The reduct of $\mathbf{B}$ to the vocabulary σ is a countable model of T and therefore it must be isomorphic to $\mathbf{A}$, because T is an ω-categorical theory. But then $\mathbf{A} \models \exists \overline{S}\neg\theta(\overline{S})$, where the symbols in $\overline{S}$ are interpreted by the additional predicates in $\mathbf{B}$. ∎

Corollary 4.2: *The set of* Π^1_1 *properties that are true on* $\mathbf{A}$ *is recursively enumerable.*

The preceding lemma is not true for arbitrary Σ^1_1 properties. As a matter of fact, there are natural Σ^1_1 sentences which are true on the random structure $\mathbf{A}$, but have probability different from 1 on C or have no probability at all. For example, the statement "there is a function f which is $1-1$ but not onto" is true on $\mathbf{A}$, but fails on every finite structure. Similarly, the statement "there is a permutation of order 2" is also true on $\mathbf{A}$, but has no asymptotic probability, because on finite structures it defines the query "there is an even number of elements". In contrast to these we have:

Lemma 4.3: *Let* $\exists \overline{S}\exists \overline{x}\forall \overline{y}\theta(\overline{S}, \overline{x}, \overline{y})$ *be a strict* Σ^1_1 *sentence that is true on* $\mathbf{A}$*. Then there is a first order sentence* ψ *over* σ *such that*

$$\mu(\psi) = 1$$

and

$$\models_{FIN} \psi \to \exists \overline{S}\exists \overline{x}\forall \overline{y}\theta(\overline{S}, \overline{x}, \overline{y}).$$

In particular, every strict Σ^1_1 *sentence that is true on* $\mathbf{A}$ *has probability 1 on* C*.*

Proof: Let $\overline{a} = (a_1, ..., a_n)$ be a sequence of elements of $\mathbf{A}$ which witness the first order existential quantifiers $\overline{x}$ in $\mathbf{A}$. Let $\mathbf{A_0}$ be the finite substructure of $\mathbf{A}$ with universe $\{a_1, ..., a_n\}$. Then there is a first order sentence ψ which is the conjunction of a finite number of the extension axioms and has the property that any model of it contains a substructure isomorphic to $\mathbf{A_0}$. Now assume that $\mathbf{B}$ is a finite model of ψ. Using the extension axioms we can find a substructure $\mathbf{B}^*$ of the random structure $\mathbf{A}$ which contains $\mathbf{A_0}$ and is isomorphic to $\mathbf{B}$. Since universal statements are preserved under substructures, we conclude that

$$\mathbf{B} \models \exists \overline{S} \exists \overline{x} \forall \overline{y} \theta(\overline{S}, \overline{x}, \overline{y}),$$

where $\overline{x}$ is interpreted by $\overline{a}$ and $\overline{S}$ is interpreted by the restriction to $\mathbf{B}$ of the relations on $\mathbf{A}$ which witness the existential second order quantifiers. ∎

From Lemmas 4.1 and 4.3 we infer immediately:

Theorem 4.4: *The 0-1 law holds for strict Σ^1_1 and strict Π^1_1 properties on the class C of all finite structures over a relational vocabulary σ. Moreover, if $\mathbf{A}$ is the countable random structure and P is a strict Σ^1_1 or strict Π^1_1 property, then*

$$\mathbf{A} \models P \iff \mu(P) = 1$$

Corollary 4.5: *The "parity" query is not definable by a strict Σ^1_1 or strict Π^1_1 sentence on the class C of all finite structures.*

As a further application of the 0-1 law for strict Σ^1_1 sentences, we obtain definability results about *connectivity* and *Hamiltonicity* on the class G of finite graphs.

Theorem 4.6: *There is no strict Σ^1_1 sentence that defines connectivity on the class G of finite graphs. Consequently, the collection of strict Σ^1_1 properties on G is not closed under negation.*

Proof: Since connectivity has probability 1, if Ψ were a strict Σ^1_1 sentence defining it on G, then Ψ would be true on the random countable graph $\mathbf{A}$. But then, the proof of Lemma 4.3 shows that there is a finite graph G such that every finite graph G* containing G as an induced subgraph is connected, which is impossible. Since *disconnectivity* is a strict Σ^1_1 property, the collection of strict Σ^1_1 properties on G is not closed under negation. ∎

The preceding theorem implies that strict Σ^1_1 and fixpoint logic are incomparable in expressive power, since connectivity is definable in fixpoint logic, and fixpoint logic is closed under negation. Notice also that, by Fagin's characterization of NP [Fa74], proving that the collection of Σ^1_1 is not closed under negation amounts to establishing that NP $\neq$ co-NP.

Theorem 4.7: *There is no strict Σ^1_1 or strict Π^1_1 sentence that defines Hamiltonicity on the class G of finite graphs.*

Proof: We sketch the proof for strict Π^1_1, the argument for strict Σ^1_1 being similar to the one for connectivity above. Assume that "Hamiltonicity" is definable by a strict Π^1_1 sentence Φ on G. Since $\mu(Hamiltonicity) = 1$ [Po76], Theorem 4.4 above implies that $\mathbf{A} \models \Phi$. From

Lemma 4.1 we infer that there is a first order sentence ψ such that $\mu(\psi) = 1$ and $\models \psi \rightarrow \Phi$. In particular, $\models_{FIN} \psi \rightarrow$ "*Hamiltonicity*". This, however, violates a theorem in Blass and Harary [BH79] to the effect that there is no first order property that has probability 1 and is a sufficient condition for "Hamiltonicity" on finite graphs. ∎

Notice that Hamiltonicity is definable by a Σ^1_1 sentence on G. Also, again by Fagin's characterization of NP [Fa74], proving that there is *no* Π^1_1 definition of Hamiltonicity on G would imply that NP $\neq$ co-NP.

5 The Decision Problem for Strict Σ^1_1 Properties

In this section we establish that the decision problem for the probabilities of strict Σ^1_1 properties is solvable and determine its computational complexity. For first order, fixpoint, and iterative logic, the proof of the 0-1 law implied immediately the solvability of the associated decision problem, although extra work was required in each case in order to pinpoint the complexity of the problem. In contrast, the only information which we can extract from the 0-1 law for strict Σ^1_1 is that the set of strict Σ^1_1 properties having probability 0 is recursively enumerable (by Corollary 4.2). We first prove here that satisfiability of strict Σ^1_1 properties on $\mathbf{A}$ is equivalent to the existence of certain *canonical* models. For simplicity we present the argument for strict Σ^1_1 sentences in which the quantifier prefix consists of second order existential variables followed by first order universal variables, i.e. for senteces of the form

$$\exists S_1 ... \exists S_l \forall y_1 ... \forall y_m \theta(S_1, ..., S_l, y_1, ..., y_m).$$

Assume that the vocabulary σ consists of a sequence $\overline{R} = (R_i, i \in I)$ of relation variables R_i. If B is a set and, for each $i \in I$, R^B_i is a relation on B of the same arity as that of R_i, then we write $\overline{R}^B$ for the sequence $(R^B_i, i \in I)$. Let $<$ be a new binary relation symbol and consider structures $\mathbf{B} = (B, \overline{R}^B, <^B)$ in which $<^B$ is a *total ordering*. Let k be a positive integer. We say that $\mathbf{B}$ is *k-rich* if for any structure $\mathbf{D}$ with k elements over the expanded vocabulary $\overline{R} \cup \{<\}$ (where $<$ is interpreted by a total ordering) there is a substructure $\mathbf{B}^*$ of $\mathbf{B}$ which is isomorphic to $\mathbf{D}$. Notice that the isomorphism takes into account *both* the relations R^D_i and the total ordering $<^D$.

Assume that S^B is an n-ary relation on B. We say that S^B is *canonical for the structure* $\mathbf{B} = (B, \overline{R}^B, <^B)$ if for any sequence $\mathbf{b} = (b_1, ..., b_n)$ from B the truth value of $S^B(b_1, ..., b_n)$ depends *only* on the isomorphism type of the substructure of $\mathbf{B}$ with universe $\{b_1, ..., b_n\}$. An expanded structure $\mathbf{B}^* = (B, \overline{R}^B, <^B, S^B_1, ..., S^B_l)$ is *canonical* if every relation S^B_i, $1 \leq i \leq l$, is canonical on $\mathbf{B}$. The intuition behind canonical structures is that the relations S^B_i are determined completely by the $(\overline{R}, <)$-types.

We say that a sequence of expanded structures $\mathbf{B}^*_\mathbf{j} = (B_j, \overline{R}^{B_j}, <^{B_j}, S^{B_j}_1, ..., S^{B^j}_l)$, $1 \leq j \leq M$, is *canonical* if all the relations $S^{B_j}_i$ are determined completely by the $(\overline{R}, <)$ types uniformly for all the structures in the sequence. Notice that every member of canonical sequence is itself a canonical structure. We can state now the main technical result of this section.

Theorem 5.1: *Let* $\mathbf{A}$ *be the random structure over the vocabulary* σ *and let* Ψ *be a strict*

Σ^1_1 *sentence of the form*

$$\exists S_1...\exists S_l \forall y_1...\forall y_m \theta(S_1,...,S_l,y_1,...,y_m).$$

Then the following are equivalent:

(1) $\mathbf{A} \models \Psi$.

(2) There is a finite canonical structure $\mathbf{B}^* = (B, \overline{R}^B, <^B, S_1^B, ..., S_l^B)$ *which is k-rich for every k*$\leq$ *m and such that*

$$\mathbf{B}^* \models \forall y_1...\forall y_m \theta(S_1,...,S_l,y_1,...,y_m).$$

.

(3) There is a canonical sequence of structures $\mathbf{B_j^*} = (B_j, \overline{R}^{B_j}, <^{B_j}, S_1^{B_j}, ..., S_l^{B^j})$*,* $1 \leq j \leq M$*, where* $M = 2^{km^d}$*,* $k =$ *the number of predicates in the vocabulary* $\overline{R} \cup \{<\}$*,* $d =$ *maximum arity of the relations in this vocabulary, such that the following hold:*

- *Each structure* $\mathbf{B_j^*}$ *has* m *elements.*
- *Every structure with* m *elements over the vocabulary* $\overline{R} \cup \{<\}$ *is isomorphic to the restriction* $\mathbf{B_j} = (B_j, \overline{R}^{B_j}, <^{B_j})$ *of some structure* $\mathbf{B_j^*}$ *in the sequence.*
- *For every* j *with* $1 \leq j \leq M$*.*

$$\mathbf{B_j^*} \models \forall y_1...\forall y_m \theta(S_1,...,S_l,y_1,...,y_m)$$

In showing that (1) $\Longrightarrow$ (2) we will use certain Ramsey type theorems which were proved by [NR77, NR83] and independently by [AH78]. We follow here the notation and terminology of [AH78] in stating these combinatorial results.

If X is any set and j is an integer, then $[X]^j$ is the collection of all subsets of X with j elements and $[X]^{\leq n} = \bigcup_{j \leq n} [X]^j$. A *system of colors* is a sequence $\mathbf{K} = (K_1, ..., K_n)$ of finite non-empty sets. A $\mathbf{K}$-*colored set* consists of a finite set X, a (total) ordering $<^X$ on X and a function $f : [X]^{\leq n} \mapsto K_1 \cup ... \cup K_n$ such that $f(Z) \in K_j$ for any $Z \in [X]^j$ and any $j \leq n$.

It is clear that every $\mathbf{K}$-colored set is isomorphic to a unique $\mathbf{K}$-*pattern*, which is a $\mathbf{K}$-colored set whose underlying set is an integer. If e, M are integers, $\mathbf{K}$ is a system of colors, P, Q are $\mathbf{K}$-patterns, then

$$Q \hookrightarrow (P)^{\leq e}_M$$

means that for every $\mathbf{K}$-colored set (X, f) of pattern Q and every partition $F : [X]^{\leq e} \mapsto M$ there is a subset Y of X such that $(Y, f\mid_Y)$ is of pattern P, and Y is *conditionally monochromatic* for F as $\mathbf{K}$-colored set, i.e. for $Z \in [Y]^j$ the value $F(Z)$ depends *only* on the $\mathbf{K}$-pattern of $(Z, f\mid_Z)$.

By iterated applications of Theorem 2.2 in [AH78] we can derive the following generalization of the classical Ramsey theorem [Ra28]:

Theorem 5.2: *For any integers* e, M*, any system of colors* $\mathbf{K}$*, and any* $\mathbf{K}$*-pattern* P*, there is a* $\mathbf{K}$*-pattern* Q *such that*

$$Q \hookrightarrow (P)^{\leq e}_M.$$

With every finite vocabulary $\sigma = \overline{R}$ in which the maximum arity is n we can associate a system of colors $\mathbf{K}$ such that every finite structure $\mathbf{B} = (B, \overline{R}^B, <^B)$, where $<^B$ is a total ordering on B, can be *coded* by a $\mathbf{K}$-colored set $(B, <^B, f)$ with $f : [B]^{\leq n} \mapsto K_1 \cup ... \cup K_n$. For example, if σ consists of a single binary relation R, then K_1 has 2 elements, K_2 has 4 elements, and $f : [B]^{\leq 2} \mapsto K_1 \cup K_2$ is such that the value $f(\{x\})$ depends only on the truth value of $R^B(x,x)$, while the value $f(\{x,y\})$ depends only on the truth values of $R^B(min(x,y), max(x,y))$ and $R^B(max(x,y), min(x,y))$. Conversely, from any such $\mathbf{K}$-pattern we can *decode* a finite structure $\mathbf{B}$.

We now have all the combinatorial machinery needed to outline the ideas in the proof of Theorem 5.1.

Sketch of Proof of Theorem 5.1.

(1) $\Longrightarrow$ (2) Let Ψ be a strict Σ^1_1 sentence Ψ such that $\mathbf{A} \models \Psi$ and assume for simplicity that Ψ has only one second order existential variable S which is ternary. We use the ternary relation S^A witnessing S on $\mathbf{A}$ to partition $[A]^{\leq 3}$ according to the *pure* S-type of a set $Z \in [A]^{\leq 3}$. This means that A is partitioned into two pieces defined by the truth value of $S^A(x,x,x)$, $[A]^2$ into two $2^{2^3-2} = 64$ pieces defined by the truth values of $S^A(min(x,y), max(x,y), max(x,y))$, etc., and finally $[A]^3$ is partitioned into $2^{3!} = 64$ pieces defined by the truth values of $S^A(min(x,y,z), mid(x,y,z), max(x,y,z))$, etc..

Let $\mathbf{B} = (B, \overline{R}^B, <^B)$ be a finite structure which is k-rich for every $k \leq m$ and let P be a $\mathbf{K}$-pattern which codes $\mathbf{B}$ in the way described above. We apply now Theorem 5.2 for $e = 3$, $M = 2 + 64 + 64 = 130$, and for P coding $\mathbf{B}$. Let Q be a $\mathbf{K}$-pattern such that $Q \hookrightarrow (P)^{\leq 3}_M$ and let $\mathbf{C}$ be the structure coded by Q. Using the extension axioms for the random structure $\mathbf{A}$ we can find in $\mathbf{A}$ a substructure $\mathbf{C_1}$ isomorphic to $\mathbf{C}$. But now Theorem 5.2 guarantees that $\mathbf{C_1}$ contains a substructure $\mathbf{B_1}$ which is isomorphic to $\mathbf{B}$ and is conditionally monochromatic as a $\mathbf{K}$-colored pattern. The structure $\mathbf{B_1}$ is k-rich for every $k \leq m$ and by taking the restriction of S^A on B_1 we can expand $\mathbf{B_1}$ to a canonical model $\mathbf{B}^*$ of $\forall y_1 ... \forall y_m \theta(S, y_1, ..., y_m)$, since universal sentences are preserved under substructures.

(2) $\Longrightarrow$ (3) This follows easily from the definitions.

(3) $\Longrightarrow$ (1) From any canonical sequence having the properties in (3) we can build a relation S^A on $\mathbf{A}$ witnessing the second order existential quantifier in Ψ by assigning tuples to S^A according to their $(\overline{R}, <)$-type, as determined by the canonical sequence. ∎

The equivalence between (1) and (3) in Theorem 5.1 implies that the set of strict Σ^1_1 properties having probability 1 is recursive. Thus, we have established that the decision problem for the probabilities of strict Σ^1_1 properties is solvable. Moreover, by computing the size of a string coding a canonical sequence that has the properties in (3), we can show that this decision problem is solvable in NEXPTIME for bounded vocabularies, and 2NEXTIME for unbounded vocabularies. These complexity bounds turn out to be tight; the NEXPTIME lower bound for bounded vocabularies follows from [Le80], while the 2NEXPTIME lower bound for unbounded vocabularies can be obtained by a generic reduction from nondeterministic Turing machines that run in doubly exponential time.

Theorem 5.3: *The decision problem for the probabilities of strict Σ^1_1 properties is NEXPTIME-complete for bounded vocabularies and 2NEXPTIME for unbounded vocabularies.*

The details of the proof will appear in the full paper. We observe that our results reveal that, in terms of the complexity of the decision problem for the probabilities, strict Σ^1_1 lies between fixpoint logic and iterative logic. Moreover, as in Section 3, there is again an exponential gap in complexity between bounded and unbounded vocabularies.

6 Concluding Remarks

We have established a 0-1 law for two logics, iterative logic, which is is essentially an infinitary first-order logic, and strict Σ^1_1, which is a fragment of second order logic. We have also proved that the decision problem for the probabilities for these logics is solvable, and have pinpointed the complexity of the problem.

Our results for strict Σ^1_1 suggest an intriguing connection between the solvability of the finite satisfiability problem for a class Ψ of first-order sentences and the 0-1 law for the second order class $\Sigma^1_1(\Psi)$. Recall that strict Σ^1_1 is $\Sigma^1_1(\Psi)$, where Ψ is the Bernays-Schönfinkel class of first order formulas. There are two other solvable prefix classes of first order formulas, namely the *Ackermann* class and the *Gödel* class [DG79]. The Ackermann class is the collection of all first order formulas with prefixes of the form $\exists^*\forall\exists^*$, that is, the prefix contains a single universal quantifier. The Gödel class is the collection of all first order formulas without equality with prefixes of the form $\exists^*\forall\forall\exists^*$, that is the prefix contains two universal quantifiers (the Gödel class with equality is unsolvable [Go84]). We raise here the question if the 0-1 law holds for the Σ^1_1 fragments defined by these classes, and whether or not the associated decision problem for the probabilities is solvable.

Independently of the 0-1 law, we have established the solvability of the strict Σ^1_1 theory of the countable random structure over an arbitrary vocabulary. Previous approaches to proving decidability of fragments of the second order theory of infinite structures (e.g., [Ra69]) concentrated on restricting the second order quantification to monadic predicates. We have considered here restrictions on the pattern of both first and second order quantifier prefixes, but have not limited ourselves to monadic second order logic. We believe that this approach will make it possible to obtain solvability for second order fragments of other structures, e.g., the rationals, and determine the boundary between the solvable and unsolvable fragments of second order theories.

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7 References

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